Location and domination in graphs.

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Outline

Detection devices and graphs

Graphs are used to model some problems:

detection devices located at some vertices
to detect/locate an intruder in some vertex,
...of course with a small number of detectors

- Detection: is there any intruder?
- Location: where is the intruder?

Detection devices and graphs

Graphs are used to model some problems:

detection devices located at some vertices

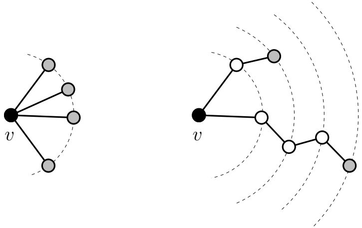
to detect/locate an intruder in some vertex,

...of course with a small number of detectors

- Detection: is there any intruder?
  Dominating sets
- Location: where is the intruder?
  Locating sets

Restrictions on detection devices

- Detect if there is an intruder in its neighborhood $\rightarrow$ 0, 1
- Detect it there is an intruder at distance $\leq k$ $\rightarrow$ 0, 1
- Detect if there is an intruder at distance $= k$ $\rightarrow$ k



Restrictions on detection devices

- At most one detection device at a vertex
- One or more detection devices at a vertex

- Detect an intruder located at any vertex of the graph
- Detect an intruder in a subset of vertices

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Questions

- Bounds relating different parameters
- Bounds relating order, size, diameter, Δ, δ,...
- Values on some families: complete graphs, paths, cycles, wheels, bipartite graphs, trees,...
- Extremal values
- Realization type results
- Graph operations: Cartesian product, strong product, complement,...
- Nordhaus-Gaddum type bounds: $p(G) + p(\overline{G})$

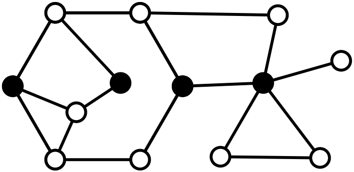
Graphs

$G = (V, E)$ graph,

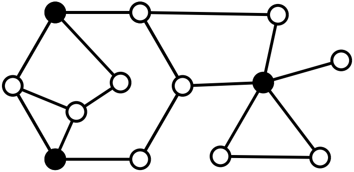
- $\overline{G}$, *complement* of G
- $N(v) = \{u : uv \in E\}$, *open neighborhood*
- $N[v] = \{v\} \cup N(v)$, *closed neighborhood*
- *true-twin vertices*: $u, v \in V$ such that $N[u] = N[v]$
- *false-twin vertices*: $u, v \in V$ such that $N(u) = N(v)$

Domination

- *Dominating set* of G, $S \subseteq V$:
  for all $v \in V \setminus S$, $S \cap N(v) \neq \emptyset$
- *Domination number* of G, $\gamma(G)$:
  minimum size of a dominating set of G



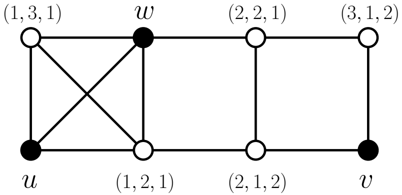
Dominating set



$\gamma(G) = 3$

Location [Slater (1975), Harary and Melter (1976)]

- *Locating set/Resolving set* of G, $S \subseteq V$:
  every vertex is uniquely determined by its vector of distances to the vertices of S
- *Location number/Metric dimension* of G, $\beta(G)$:
  minimum size of a locating set of G
- *Locating code/Metric basis* of G:
  locating and dominating set of minimum size



Location and domination [Henning and Oellermann, 2004)

- *Locating and dominating set (MLD-set)* of G, $S \subseteq V$:
  - dominating set of G
  - locating set of G

- *Location and domination number*, $\eta(G)$:
  minimum size of a locating and dominating set of G

- *Locating and dominating code* of G:
  locating and dominating set of minimum size

$$\boxed{\max\{\gamma(G), \beta(G)\} \leq \eta(G) \leq \gamma(G) + \beta(G)}$$

Location-domination [Slater, 1988]

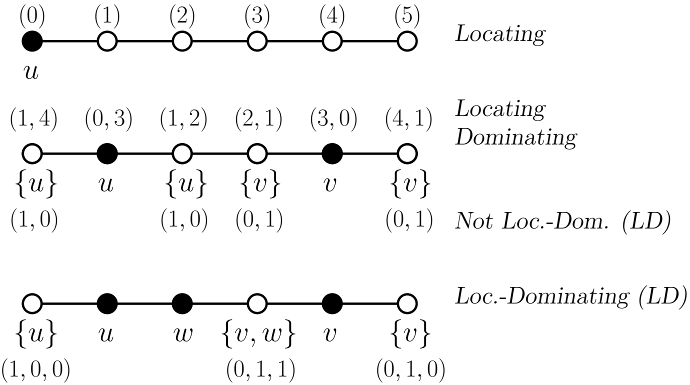
- *Locating-dominating set (LD-set)* of G, $S \subseteq V$:
  - dominating set of G
  - $N_G(u) \cap S \neq N_G(v) \cap S$, if $u, v \in V \setminus S$, $u \neq v$
- *Location-domination number*, $\lambda(G)$:
  minimum size of a locating-dominating set of G
- *Locating-dominating code (LD-code)* of G:
  LD-set of minimum size

Location-Domination

- S LD-set $\Rightarrow$ S dominating set
- S LD-set $\Rightarrow$ S locating set
  - $\gamma(G) \leq \lambda(G)$
  - $\beta(G) \leq \lambda(G)$

$$\boxed{\max\{\gamma(G), \beta(G)\} \leq \eta(G) \leq \lambda(G)}$$

Location and Domination/Location-Domination



Location and Domination/Location-Domination

$$S = \{u_1, \ldots, u_r\}$$

$$x \in V \setminus S \quad \longrightarrow \quad \ell(x) = (x_1, \ldots, x_r)$$

Param.	x_i	Conditions
γ	$\begin{cases} 1, & \text{if } x \in N(u_i); \\ 0, & \text{otherwise.} \end{cases}$	$\exists x_i = 1$
β	$d(x, u_i)$	$\ell(x) \neq \ell(y)$, if $x \neq y$
η	$d(x, u_i)$	$\exists x_i = 1$ $\quad \ell(x) \neq \ell(y)$, if $x \neq y$
λ	$\begin{cases} 1, & \text{if } x \in N(u_i); \\ 0, & \text{otherwise.} \end{cases}$	$\exists x_i = 1$ $\quad \ell(x) \neq \ell(y)$, if $x \neq y$

Identifying codes [Karpovsky, Chakrabarty and Levitin, 1998]

- *Identifying set* of G, $S \subseteq V$:
  - dominating set of G
  - $N_G[u] \cap S \neq N_G[v] \cap S, \quad$ if $u, v \in V$, $u \neq v$
- *Identifying number*, $\iota(G)$:
  minimum size of an identifying set of G
- *Identifying code* of G:
  identifying set of minimum size

Identifying codes exist only in true-twin free graphs

$$\boxed{\max\{\gamma(G), \beta(G)\} \leq \eta(G) \leq \lambda(G) \leq \iota(G)}$$

Watching systems [Auger, Charon, Hudry and Lobstein, 2013]

- *watcher*: $w_i = (v_i, A_i)$, where $w_i = (v_i, A_i)$, $v_i \in V$, $A_i \subseteq N[v_i]$
- *label* of $u \in V$: $L(u) = \{w_i : u \in A_i\}$
- *watching system*: $S = \{w_i : i \in I\}$ such that
  $L(u) \neq \emptyset$ for all $u \in V$
  $L(u) \neq L(v)$ if $u \neq v$
- *watching number*, $\omega(G)$:
  minimum size of a watching system set of G

Watching systems exist in all graphs

$$\boxed{\omega(G) \leq \gamma(G) \lceil \log_2(\Delta + 2) \rceil}$$

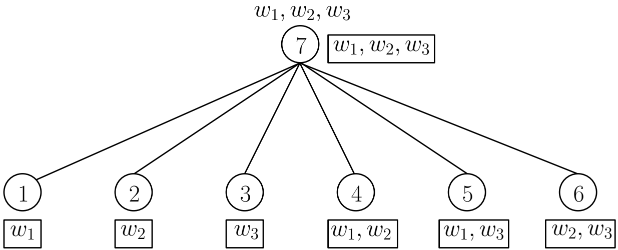
$$\boxed{\omega(G) \leq \iota(G) \text{ if } G \text{ is twin-free}}$$

Watching number: example

$G = K_{1,6}$: $i(G) = 6$, $\omega(G) = 3$

$W = \{w_1, w_2, w_3\}$, $l(w_i) = 7$

$A(w_1) = \{1, 4, 5, 7\}$, $A(w_2) = \{2, 4, 6, 7\}$, $A(w_3) = \{3, 5, 6, 7\}$



Some families

	P_n $(n\geq4)$	C_n $(n\geq3)$	K_n $(n\geq2)$	$K_{1,n-1}$ $(n\geq3)$	$K_{r,s}$ $(2\leq r\leq s)$
$\gamma(G)$	$\lceil \frac{n}{3} \rceil$	$\lceil \frac{n}{3} \rceil$	1	1	2
$\beta(G)$	1	2	$n-1$	$n-2$	$n-2$
$\eta(G)$	$\lceil \frac{n}{3} \rceil$	$\lceil \frac{n}{3} \rceil$	$n-1$	$n-1$	$n-2$
$\lambda(G)$	$\lceil \frac{2n}{5} \rceil$	$\lceil \frac{2n}{5} \rceil$	$n-1$	$n-1$	$n-2$
	P_n $(n\geq4)$	C_n $(n\geq7)$	K_n $(n\geq2)$	$K_{1,n-1}$ $(n\geq3)$	$K_{r,s}$ $(2\leq r\leq s)$
$\iota(G)$	$\lceil \frac{n+1}{2} \rceil$	$3\lceil \frac{n}{2} \rceil - n$	—	$n-1$	$n-2$
$\omega(G)$	$\lceil \frac{n+1}{2} \rceil$	$\lceil \frac{n}{2} \rceil$	$\lceil \log_2(n+1) \rceil$	$\lceil \log_2(n+1) \rceil$	(*)

(*) [Hernando, Mora and Pelayo, 2012]

Bounds

$$G = (V, E),\ |V| = n,\ diam(G) = D \geq 3,$$

- $$\beta + D \leq n \leq \left(\left\lfloor \frac{2D}{3} \right\rfloor + 1 \right)^{\beta} + \beta \sum_{i=1}^{\lceil D/3 \rceil} (2i-1)^{\beta-1}$$
  [Hernando, Mora, Pelayo, Seara and Wood, 2010]
- $$\eta + \left\lceil \frac{2D}{3} \right\rceil \leq n \leq \eta\left(3^{\eta-1} + 1\right), \text{ if } G \ncong K_{1,n-1}$$
  [Cáceres, Hernando, Mora, Pelayo and Puertas, 2013]
- $$\lambda + \left\lceil \frac{3D-1}{5} \right\rceil \leq n \leq \lambda + 2^{\lambda} - 1$$
  [Cáceres, Hernando, Mora, Pelayo and Puertas, 2013]

All bounds are tight!

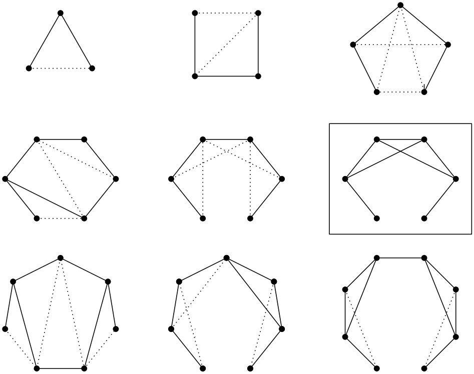
Bounds

$$G = (V, E), |V| = n$$

- $\iota + 1 \leq n \leq 2^{\iota} - 1$, if G is true-twin free
  [Karpovsky, Chakrabarty and Levitin, 1998]
- $n \leq 2^{\omega} - 1$
  [Auger, Charon, Hudry and Lobstein, 2010]
- $\omega(G) \leq \frac{2n}{3}$, if G is a connected graph of order 3 or ≥ 5
  [Auger, Charon, Hudry and Lobstein, 2010]

Small values of $\eta(G)$ and $\lambda(G)$

- There are 51 non isomorphic graphs satisfying $\eta(G) = 2$
- There are 16 non isomorphic graphs satisfying $\lambda(G) = 2$ [Cáceres, Hernando, Mora, Pelayo and Puertas, 2013]



Large values of $\eta(G)$ and $\lambda(G)$

- $\eta(G) = n - 1 \Leftrightarrow G \cong K_n, K_{1,n-1}$
  [Henning and Oellermann, 2004]
- $\lambda(G) = n - 1 \Leftrightarrow G \cong K_n, K_{1,n-1}$
  [Slater, 1988]
- Graphs satisfying $\eta(G) = n - 2$
  [Henning and Oellermann, 2004]
- Graphs satisfying $\lambda(G) = n - 2$:
  $\eta(G) = n - 2 \iff \lambda(G) = n - 2$
  [Cáceres, Hernando, Mora, Pelayo and Puertas, 2013]

Realization Theorems

[Cáceres, Hernando, Mora, Pelayo and Puertas, 2013]

- G graph, $\boxed{\max\{\gamma(G), \beta(G)\} \leq \eta(G) \leq \gamma(G) + \beta(G)}$

  $a, b, c \in \mathbb{N}, \quad \max\{a, b\} \leq c \leq a + b \quad \Rightarrow$
  $\exists G$ graph satisfying $\gamma(G) = a, \beta(G) = b, \eta(G) = c$,
  except if $1 = b < a < c = a + 1$

- T tree, $|V(T)| \geq 3$, $T \not\cong P_6$ $\boxed{\eta(T) \leq \lambda(T) \leq 2\eta(T) - 2}$

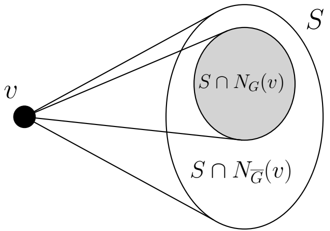
  $a, b \in \mathbb{N}, \quad 3 \leq a \leq b \leq 2a - 2 \quad \Rightarrow$
  $\exists T$ tree satisfying $\eta(T) = a$, $\lambda(T) = b$

$\lambda(G)$ versus $\lambda(\overline{G})$ [Hernando, Mora and Pelayo (2013)]

S LD-set of $G=(V,E)$:

- dominating set of G
- $N_G(u)\cap S \neq N_G(v)\cap S$, if $u,v\in V\setminus S$, $u\neq v$

- $N_G(x)\cap S \neq N_G(y)\cap S \Leftrightarrow N_{\overline{G}}(x)\cap S \neq N_{\overline{G}}(y)\cap S$



$\lambda(G)$ versus $\lambda(\overline{G})$

S LD-set of G, then

- S LD-set of $\overline{G} \Leftrightarrow S$ dominating set of $\overline{G}$.
- S LD-set of $\overline{G} \Leftrightarrow \nexists u \in V \setminus S$, u dominates S in G.
- $\exists u \in V \setminus S$, u dominates S in G
  $\Rightarrow S \cup \{u\}$ is an LD-set of $\overline{G}$

$$|\lambda(G) - \lambda(\overline{G})| \leq 1$$

Global LD-sets

- *global LD-set of G*: LD-set of G and of $\overline{G}$

  i.e., $\nexists u \in V \setminus S$, u dominates S in G.

- G contains a global LD-code $\Rightarrow \lambda(\overline{G}) \leq \lambda(G)$

Graphs satisfying $\lambda(\overline{G}) = \lambda(G) + 1$?

Global LD-sets

G contains a non-global LD-set $S \Rightarrow$

- $\exists u \in V \setminus S$, u dominates S
- $ecc(u) \leq 2$, $\mathrm{rad}(G) \leq 2$, $\mathrm{diam}(G) \leq 4$

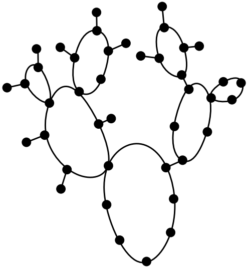
$\lambda(\overline{G}) = \lambda(G) + 1 \Rightarrow$

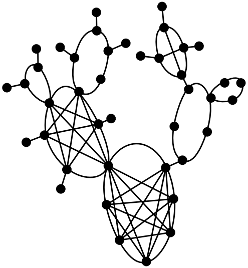
$\Rightarrow$ every LD-code of G is non-global

$\Rightarrow$ G connected and $\mathrm{diam}(G) \leq 4$

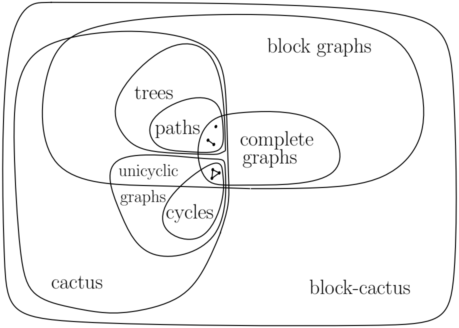
Block-cactus

- Block of $G = (V, E)$: maximally connected subgraph with no cut vertices
- Cactus: connected graph s.t. all blocks are cycles or K_2, i.e., there is no edge lying on two different cycles
- Block-cactus: connected graph s.t. all blocks are cycles or complete graphs





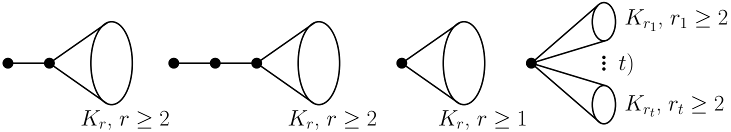
Families of block-cactus



Block-cactus

Block-cactus s.t. $\lambda(\overline{G}) = \lambda(G) + 1$

G is one of the following graphs:



- T tree of order $\geq 3 \Rightarrow \lambda(\overline{T}) \leq \lambda(T)$

Bipartite graphs

Bipartite graphs s.t. $\lambda(\overline{G}) = \lambda(G) + 1$

$$G = (V, E)\ V = V_1 \cup V_2, |V_1| = r, |V_2| = s, 2 \leq r \leq s$$

- $r = 1, 2 \Rightarrow \lambda(\overline{G}) \leq \lambda(G)$
- S LD-code of G
  - $S \cap V_1 \neq \emptyset$ and $S \cap V_2 \neq \emptyset \quad \Rightarrow \lambda(\overline{G}) \leq \lambda(G)$.
  - $S = V_2$ and $r < s \Rightarrow \lambda(\overline{G}) \leq \lambda(G)$.

Bipartite graphs satisfying $\lambda(\overline{G}) = \lambda(G) + 1$

$$G = (V, E),\ V = V_1 \cup V_2,\ |V_1| = r,\ |V_2| = s,\ 3 \leq r \leq s$$

Theorem

$$\lambda(\overline{G}) = \lambda(G) + 1 \Rightarrow \frac{3r}{2} \leq s \leq 2^r - 1$$

Proof.

$\lambda(\overline{G}) = \lambda(G) + 1 \Rightarrow V_1$ LD-code of G

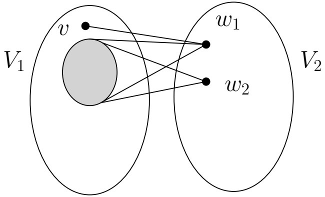
$s \leq 2^r - 1$

V_1 LD-code $\Rightarrow s \leq 2^r - 1$

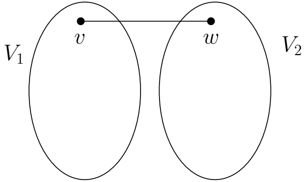
Proof of $\frac{3r}{2} \leq s$

$$\boxed{\frac{3r}{2} \leq s}$$

$\forall v \in V_1$, $V_1 \setminus \{v\}$ is not an LD-set:



$N(w_1) \oplus N(w_2) = \{v\}$



$N(w) \oplus \emptyset = \{v\}$

Proof of $\frac{3r}{2} \leq s$: the graph G^*

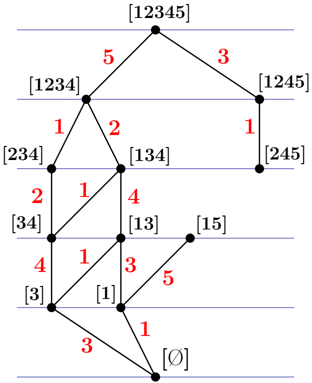
G^* edge-labeled graph associated to G:

- $V(G^*) = V_2 \cup \{w_0\}$, $w_0 \notin V_2$ and define $N(w_0) = \emptyset$
- $w_h w_k \in E(G^*) \Leftrightarrow N(w_h) \oplus N(w_k) = \{v\}$ for some $v \in V_1$
- $\ell(w_h w_k) = N(w_h) \oplus N(w_k) \in V_1$

Example of graph G^*

$V_1 = \{1, 2, 3, 4, 5\}$

$V_2 = \{[12345], [1234], [1245], [134], [234], [245], [13], [15], [34], [1], [3]\}$



Proof of $\frac{3r}{2} \leq s$ using the graph G^*

V_1 LD-code and $\nexists$ LD-code with vertices in both stable sets
$\Rightarrow G^*$ satisfies:

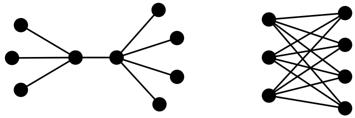
- $|V(G^*)| = s + 1$, $|E(G^*)| \geq 2r$
- G^* is bipartite
- incident edges have different labels
- walks contain an even number of edges with label v, $\forall v \in V_1$, iff they are closed
- G^* contains a subgraph H^* of size $2r$ such that all its connected components are cactus
- G with cc. cactus with no C_3, then $|E(G)| \leq \frac{4}{3}(|V(G)| - 1)$.

$$\Rightarrow s \geq \frac{3r}{2}$$

Bipartite graphs with $\lambda(\overline{G}) - \lambda(G) \in \{-1, 0, 1\}$

Given integers r, s, $3 \leq r \leq s$ there are bipartite graphs G with stable parts V_1, V_2 satisfying $|V_1| = r$, $|V_2| = s$ and:

- $\lambda(\overline{G}) = \lambda(G) - 1$: double star $K_2(r-1, s-1)$
- $\lambda(\overline{G}) = \lambda(G)$: complete bipartite graphs $K(r, s)$

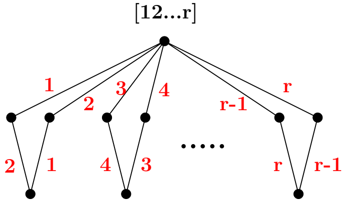


- $\lambda(\overline{G}) = \lambda(G) + 1$, if $\dfrac{3r}{2} + 1 \leq s \leq 2^r - 1$: $G(r, s)$

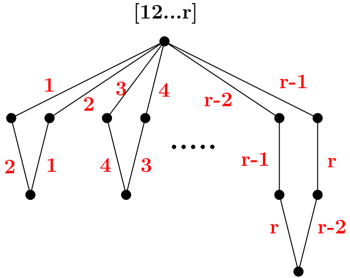
Bipartite graphs with $\lambda(\overline{G}) = \lambda(G) + 1$

- (r,s), $r,s \in \mathbb{N}$, $3 \leq r$ and $\frac{3r}{2} + 1 \leq s \leq 2^r - 1$, $\exists G(r,s)$ bipartite graph such that $\lambda(\overline{G}) > \lambda(G)$.

$V_1 = \{1, 2, \ldots, r\}$ and $s = \left\lceil \frac{3r}{2} + 1 \right\rceil$:



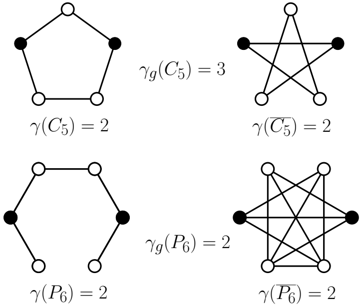
G^*, r even



G^*, r odd

Global domination [Brigham and Carrington, 1998]

- $S \subseteq V$, *global dominating set* of G:
  a dominating set of G and of $\overline{G}$
- *global domination number*, $\gamma_g(G)$:
  minimum size of a global dominating set of G



Global LD-number

- *global LD-number of G*:
  minimum size of a global LD-set

✓ $\lambda_g(G) = \lambda_g(\overline{G})$

For any graph $G = (V, E)$,

- $\lambda(G) \leq \lambda_g(G) \leq \lambda(G) + 1$
- $\lambda_g(G) = \lambda(G) \iff G$ has a global LD-code
  $\lambda_g(G) = \lambda(G) + 1 \iff$ every LD-code of G is non-global
- $\lambda(\overline{G}) \neq \lambda(G) \Rightarrow \lambda_g(G) = \max\{\lambda(G), \lambda(\overline{G})\}$
  $\lambda(\overline{G}) = \lambda(G) \Rightarrow \lambda_g(G) \in \{\lambda(G), \lambda(G) + 1\}$

Some values

	P_n (≥ 7)	C_n (≥ 7)	W_n $(n\geq 8)$	K_n $(n\geq 3)$
$\lambda(G)$	$\lceil \frac{2n}{5} \rceil$	$\lceil \frac{2n}{5} \rceil$	$\lceil \frac{2n-2}{5} \rceil$	$n-1$
$\lambda(\overline{G})$	$\lceil \frac{2n-2}{5} \rceil$	$\lceil \frac{2n-2}{5} \rceil$	$1+\lceil \frac{2n-4}{5} \rceil$	n
$\lambda_g(G)$	$\lceil \frac{2n}{5} \rceil$	$\lceil \frac{2n}{5} \rceil$	$1+\lceil \frac{2n-4}{5} \rceil$	n

	$K_{1,n-1}$ $()n\geq 4$	$K_{r,s}$ $(2\leq r\leq s)$	$K_2(r,s)$ $(2\leq r\leq s)$
$\lambda(G)$	$n-1$	$n-2$	$n-2$
$\lambda(\overline{G})$	$n-1$	$n-2$	$n-3$
$\lambda_g(G)$	$n-1$	$n-2$	$n-2$