# Some metric results on weighted 2–Cayley digraphs

#### Seminari CombGraph, MA-IV

#### F. Aguiló Gost joint with A. Miralles and M. Zaragozá

#### Dept. Matemàtica Aplicada IV Universitat Politècnica de Catalunya

2014 - 02 - 06

Seminari CombGraph, MA–IV Some metric results on weighted 2–Cayley digraphs

Minimium path diagram of weighted 2–Cayley digrafs Maximum genus of numerical 3–semigroups

#### INTRODUCTION

Seminari CombGraph, MA–IV Some metric results on weighted 2–Cayley digraphs

イロト イヨト イヨト イヨト

э.

## Introduction

Let  $G_N = \langle a_1, ..., a_n \rangle$  be a finite Abelian group (additive notation).

Seminari CombGraph, MA–IV Some metric results on weighted 2–Cayley digraphs

イロト イロト イヨト イヨト 三日

#### Introduction

Let  $G_N = \langle a_1, ..., a_n \rangle$  be a finite Abelian group (additive notation).

Consider the Cayley digraph  $\operatorname{Cay}(G_N, \{a_1, ..., a_n\}, \{W_1, ..., W_n\})$ , where  $W_i > 0$  are the weighs of the arcs  $u \xrightarrow{W_i} u + a_i$  for any  $u \in G_N$  and  $i \in \{1, ..., n\}$ .

(日) (周) (日) (日) (日)

### Introduction

Let  $G_N = \langle a_1, ..., a_n \rangle$  be a finite Abelian group (additive notation).

Consider the Cayley digraph  $\operatorname{Cay}(G_N, \{a_1, ..., a_n\}, \{W_1, ..., W_n\})$ , where  $W_i > 0$  are the weighs of the arcs  $u \xrightarrow{W_i} u + a_i$  for any  $u \in G_N$  and  $i \in \{1, ..., n\}$ .

The length of a path is the sum of the weighs of his arcs.

A minimum path from u to v is a path of minimum lentph over all paths from u to v.

The distance from u to v is the lentph of a minimum path from u to v.

#### Introduction

Many metric results use minimum distance diagrams (MDD) related to these digraphs as a main working tool.

《曰》 《聞》 《臣》 《臣》

- 20

#### Introduction

Many metric results use minimum distance diagrams (MDD) related to these digraphs as a main working tool.

These diagrams can be seen as a geometric view of the hole digraph. They can assist intuition when working with distance–related ideas and reasonings.

(日) (四) (日) (日) (日)

#### Introduction

Sabariego and Santos, 2009 gave an algebraic description of a MDD related to  $G = \text{Cay}(\mathbb{Z}_N, \{a_1, ..., a_n\}, \{W_1, ..., W_n\}).$ 

Seminari CombGraph, MA–IV Some metric results on weighted 2–Cayley digraphs

(日) (四) (日) (日) (日)

- 20

#### Introduction

Sabariego and Santos, 2009 gave an algebraic description of a MDD related to  $G = \text{Cay}(\mathbb{Z}_N, \{a_1, ..., a_n\}, \{W_1, ..., W_n\})$ . Given  $(i_1, ..., i_n) \in \mathbb{N}^n$ , set  $\llbracket i_1, ..., i_n \rrbracket = [i_1, i_1 + 1] \times \cdots \times [i_n, i_n + 1] \in \mathbb{R}^n$ ,  $\delta(i_1, ..., i_n) = i_1 W_1 + \cdots + i_n W_n$ ,  $\Delta(i_1, ..., i_n) = \{\llbracket j_1, ..., j_n \rrbracket : 0 \le j_1 \le i_1, ..., 0 \le j_n \le i_n\}$ ,  $C_m = \{\llbracket i_1, ..., i_n \rrbracket : i_1 a_1 + \cdots + i_n a_n \equiv m (\text{mod } N)\}$ .

#### Introduction

Sabariego and Santos, 2009 gave an algebraic description of a MDD related to  $G = \text{Cay}(\mathbb{Z}_N, \{a_1, ..., a_n\}, \{W_1, ..., W_n\})$ . Given  $(i_1, ..., i_n) \in \mathbb{N}^n$ , set  $[\![i_1, ..., i_n]\!] = [i_1, i_1 + 1] \times \cdots \times [i_n, i_n + 1] \in \mathbb{R}^n$ ,  $\delta(i_1, ..., i_n) = i_1 W_1 + \cdots + i_n W_n$ ,  $\Delta(i_1, ..., i_n) = i_1 W_1 + \cdots + i_n W_n$ ,  $\Delta(i_1, ..., i_n) = \{[\![j_1, ..., j_n]\!] : 0 \le j_1 \le i_1, ..., 0 \le j_n \le i_n\}$ ,  $C_m = \{[\![i_1, ..., i_n]\!] : i_1 a_1 + \cdots + i_n a_n \equiv m (\text{mod } N)\}$ . An MDD related to C is any unitary who made region

An MDD related to G is any unitary-cube-made region,  $\mathcal{H} \subset \mathbb{R}_{\geq 0}^{n}$ , such that

#### Introduction

Sabariego and Santos, 2009 gave an algebraic description of a MDD related to  $G = \text{Cay}(\mathbb{Z}_N, \{a_1, ..., a_n\}, \{W_1, ..., W_n\})$ . Given  $(i_1, ..., i_n) \in \mathbb{N}^n$ , set

$$\llbracket i_1, ..., i_n \rrbracket = [i_1, i_1 + 1] \times \dots \times [i_n, i_n + 1] \in \mathbb{R}^n,$$

$$\delta(i_1,...,i_n) = i_1 W_1 + \dots + i_n W_n,$$

$$\Delta(i_1,...,i_n) = \{ [ [j_1,...,j_n] ] : 0 \le j_1 \le i_1,..., 0 \le j_n \le i_n \},\$$

 $C_m = \{ [\![i_1, ..., i_n]\!] : i_1 a_1 + \dots + i_n a_n \equiv m (\text{mod } N) \}.$ 

An MDD related to G is any unitary-cube-made region,  $\mathcal{H} \subset \mathbb{R}_{\geq 0}^{n}$ , such that (1) Each  $[m]_{N} \in \mathbb{Z}_{N}$  is represented by a unique  $[i_{1}, ..., i_{n}] \in \mathcal{H} \cap C_{m}$ .

#### Introduction

Sabariego and Santos, 2009 gave an algebraic description of a MDD related to  $G = \text{Cay}(\mathbb{Z}_N, \{a_1, ..., a_n\}, \{W_1, ..., W_n\})$ . Given  $(i_1, ..., i_n) \in \mathbb{N}^n$ , set

$$[\![i_1, ..., i_n]\!] = [i_1, i_1 + 1] \times \cdots \times [i_n, i_n + 1] \in \mathbb{R}^n,$$

$$\delta(i_1,...,i_n) = i_1 W_1 + \cdots + i_n W_n,$$

$$\Delta(i_1,...,i_n) = \{ [ j_1,...,j_n ] : 0 \le j_1 \le i_1,..., 0 \le j_n \le i_n \},\$$

 $C_m = \{ [[i_1, ..., i_n]] : i_1 a_1 + \dots + i_n a_n \equiv m \pmod{N} \}.$ 

An MDD related to G is any unitary-cube-made region,  $\mathcal{H} \subset \mathbb{R}_{\geq 0}^{n}$ , such that

(1) Each  $[m]_N \in \mathbb{Z}_N$  is represented by a unique  $[i_1, ..., i_n] \in \mathcal{H} \cap C_m$ .

(2) If 
$$[m]_N \sim [\![i_1, ..., i_n]\!]$$
, then  
 $\delta(i_1, ..., i_n) = \min\{\delta(j_1, ..., j_n) : [\![j_1, ..., j_n]\!] \in C_m\}.$ 

< ロト ( 母 ) ( ヨ ) ( 1 )

#### Introduction

Sabariego and Santos, 2009 gave an algebraic description of a MDD related to  $G = \text{Cay}(\mathbb{Z}_N, \{a_1, ..., a_n\}, \{W_1, ..., W_n\})$ . Given  $(i_1, ..., i_n) \in \mathbb{N}^n$ , set

$$[\![i_1, ..., i_n]\!] = [i_1, i_1 + 1] \times \cdots \times [i_n, i_n + 1] \in \mathbb{R}^n,$$

$$\delta(i_1,...,i_n) = i_1 W_1 + \cdots + i_n W_n,$$

$$\Delta(i_1, ..., i_n) = \{ [ [j_1, ..., j_n] ] : 0 \le j_1 \le i_1, ..., 0 \le j_n \le i_n \},\$$

 $C_m = \{ [\![i_1, ..., i_n]\!] : i_1 a_1 + \dots + i_n a_n \equiv m (\text{mod } N) \}.$ 

An MDD related to G is any unitary-cube-made region,  $\mathcal{H} \subset \mathbb{R}_{\geq 0}^{n}$ , such that

- (1) Each  $[m]_N \in \mathbb{Z}_N$  is represented by a unique  $[i_1, ..., i_n] \in \mathcal{H} \cap C_m$ .
- (2) If  $[m]_N \sim [\![i_1, ..., i_n]\!]$ , then  $\delta(i_1, ..., i_n) = \min\{\delta(j_1, ..., j_n) : [\![j_1, ..., j_n]\!] \in C_m\}.$
- (3) If  $[\![i_1, ..., i_n]\!] \in \mathcal{H}$ , then  $\Delta(i_1, ..., i_n) \subseteq \mathcal{H}$ .

Seminari CombGraph, MA–IV

Some metric results on weighted 2–Cayley digraphs

### Introduction

Exemple 1:  $Cay(\mathbb{Z}_2 \times \mathbb{Z}_6, \{(1,2), (1,1)\}, \{1,1\})$ 

Seminari CombGraph, MA–IV Some metric results on weighted 2–Cayley digraphs

《曰》 《聞》 《臣》 《臣》

э.

#### Introduction

Exemple 1:  $Cay(\mathbb{Z}_2 \times \mathbb{Z}_6, \{(1,2), (1,1)\}, \{1,1\})$ 



<ロト <問ト < 回ト < 回ト

э

### Introduction

#### Exemple 1: $Cay(\mathbb{Z}_2 \times \mathbb{Z}_6, \{(1,2), (1,1)\}, \{1,1\})$



Seminari CombGraph, MA–IV Some metric results on weighted 2–Cayley digraphs

<ロト <問ト < 回ト < 回ト

э

### Introduction

#### Exemple 1: $Cay(\mathbb{Z}_2 \times \mathbb{Z}_6, \{(1,2), (1,1)\}, \{1,1\})$



Seminari CombGraph, MA–IV Some metric results on weighted 2–Cayley digraphs

<ロト <問ト < 回ト < 回ト

#### Introduction

#### Exemple 1: $Cay(\mathbb{Z}_2 \times \mathbb{Z}_6, \{(1,2), (1,1)\}, \{1,1\})$



Seminari CombGraph, MA–IV Some metric results on weighted 2–Cayley digraphs

<ロト <問ト < 回ト < 回ト

Minimium path diagram of weighted 2–Cayley digrafs Maximum genus of numerical 3–semigroups

## Introduction

Example 3:  $Cay(\mathbb{Z}_9, \{1, 4, 7\}, \{1, 1, 1\})$ 

Seminari CombGraph, MA–IV Some metric results on weighted 2–Cayley digraphs

<ロ> (日) (日) (日) (日) (日)

э

## Introduction

Example 3:  $Cay(\mathbb{Z}_9, \{1, 4, 7\}, \{1, 1, 1\})$ 



<ロト <問ト < 回ト < 回ト

## Introduction

Example 3:  $Cay(\mathbb{Z}_9, \{1, 4, 7\}, \{1, 1, 1\})$ 





Seminari CombGraph, MA–IV Some metric results on weighted 2–Cayley digraphs

## Introduction

Example 3:  $Cay(\mathbb{Z}_9, \{1, 4, 7\}, \{1, 1, 1\})$ 







Seminari CombGraph, MA–IV Some metric results on weighted 2–Cayley digraphs

## Introduction

Example 3:  $Cay(\mathbb{Z}_9, \{1, 4, 7\}, \{1, 1, 1\})$ 







(日) (四) (日) (日)



## Introduction

Example 3:  $Cay(\mathbb{Z}_9, \{1, 4, 7\}, \{1, 1, 1\})$ 







<ロト <問ト < 回ト < 回ト





## Introduction

Example 3:  $Cay(\mathbb{Z}_9, \{1, 4, 7\}, \{1, 1, 1\})$ 













<ロト <問ト < 回ト < 回ト

Seminari CombGraph, MA–IV Some metric results on weighted 2–Cayley digraphs

## Introduction

Example 3:  $Cay(\mathbb{Z}_9, \{1, 4, 7\}, \{1, 1, 1\})$ 













<ロト < 四ト < 回ト < 回ト



## Introduction

Example 3:  $Cay(\mathbb{Z}_9, \{1, 4, 7\}, \{1, 1, 1\})$ 

















 $\langle \Box \rangle \langle \overline{C} \rangle \langle \overline{C} \rangle \langle \overline{C} \rangle \langle \overline{C} \rangle \rangle$ Some metric results on weighted 2–Cayley digraphs

Seminari CombGraph, MA–IV

## Introduction

Example 3:  $Cay(\mathbb{Z}_9, \{1, 4, 7\}, \{1, 1, 1\})$ 



















Seminari Comb<br/>Graph,  $\rm MA{-}IV$ 

Some metric results on weighted 2–Cayley digraphs

## Introduction

Many shapes appear in the 3D case

Seminari CombGraph, MA–IV Some metric results on weighted 2–Cayley digraphs

э.

## Introduction

Many shapes appear in the 3D case



## Introduction

Many shapes appear in the 3D case



<ロト <回ト < 回ト < 三

## Introduction

Many shapes appear in the 3D case



Minimium path diagram of weighted 2–Cayley digrafs Maximum genus of numerical 3–semigroups

#### Introduction



Seminari CombGraph, MA–IV Some metric results on weighted 2–Cayley digraphs

イロト イヨト イヨト イヨト

Minimium path diagram of weighted 2–Cayley digrafs Maximum genus of numerical 3–semigroups

#### Introduction



Seminari CombGraph, MA–IV Some metric results on weighted 2–Cayley digraphs

イロト イヨト イヨト イヨト

Minimium path diagram of weighted 2–Cayley digrafs Maximum genus of numerical 3–semigroups

#### Introduction



Seminari CombGraph, MA–IV Some metric results on weighted 2–Cayley digraphs

イロト イヨト イヨト イヨト

Minimium path diagram of weighted 2–Cayley digrafs Maximum genus of numerical 3–semigroups

#### Introduction



Seminari CombGraph, MA–IV Some metric results on weighted 2–Cayley digraphs
Minimium path diagram of weighted 2–Cayley digrafs Maximum genus of numerical 3-semigroups

# Introduction



Some metric results on weighted 2-Cayley digraphs

Minimium path diagram of weighted 2–Cayley digrafs Maximum genus of numerical 3–semigroups

# Introduction



Minimium path diagram of weighted 2–Cayley digrafs Maximum genus of numerical 3–semigroups

# Introduction



Minimium path diagram of weighted 2–Cayley digrafs Maximum genus of numerical 3–semigroups

# Introduction



# Introduction

Most properties of MDDs are well known when n = 2. And many studies on the diameter have been derived from their geometrical approach (sharp lower bound, optimal-diameter families of digraphs, etc).

< ロト ( 母 ) ( ヨ ) ( コ ) ( コ ) ( 1 )

# Introduction

Most properties of MDDs are well known when n = 2. And many studies on the diameter have been derived from their geometrical approach (sharp lower bound, optimal-diameter families of digraphs, etc).

Less is known on generic results of MDDs when  $n \ge 3$ . There is no geometrical description of them.

< ロト ( 母 ) ( ヨ ) ( コ ) ( コ ) ( 1 )

### Introduction

Given  $a_1, ..., a_n \in \mathbb{N}$ , with  $1 < a_1 < ... < a_n$  and  $gcd(a_1, ..., a_n) = 1$ , the numerical n-semigroup S generated by  $\{a_1, ..., a_n\}$  is

 $S = \langle a_1, ..., a_n \rangle = \{ x_1 a_1 + \dots + x_n a_n : x_1, ..., x_n \in \mathbb{N} \}.$ 

▲日▼ ▲母▼ ▲日▼ ▲日▼ ヨー シタク

### Introduction

Given  $a_1, ..., a_n \in \mathbb{N}$ , with  $1 < a_1 < ... < a_n$  and  $gcd(a_1, ..., a_n) = 1$ , the numerical n-semigroup S generated by  $\{a_1, ..., a_n\}$  is

$$S = \langle a_1, ..., a_n \rangle = \{ x_1 a_1 + \dots + x_n a_n : x_1, ..., x_n \in \mathbb{N} \}.$$

The set of holes of S is  $\overline{S} = \mathbb{N} \setminus S$  and  $|\overline{S}| < \infty$  holds.

### Introduction

Given  $a_1, ..., a_n \in \mathbb{N}$ , with  $1 < a_1 < ... < a_n$  and  $gcd(a_1, ..., a_n) = 1$ , the numerical n-semigroup S generated by  $\{a_1, ..., a_n\}$  is

$$S = \langle a_1, ..., a_n \rangle = \{ x_1 a_1 + \dots + x_n a_n : x_1, ..., x_n \in \mathbb{N} \}.$$

The set of holes of S is  $\overline{S} = \mathbb{N} \setminus S$  and  $|\overline{S}| < \infty$  holds.

The Frobenius number and the genus of S are  $\mathfrak{f}(S) = \max \overline{S}$  and  $\mathfrak{g}(S) = |\overline{S}|$ , respectively.

▲日▼ ▲母▼ ▲日▼ ▲日▼ ヨー シタク

### Introduction

Given  $a_1, ..., a_n \in \mathbb{N}$ , with  $1 < a_1 < ... < a_n$  and  $gcd(a_1, ..., a_n) = 1$ , the numerical n-semigroup S generated by  $\{a_1, ..., a_n\}$  is

$$S = \langle a_1, ..., a_n \rangle = \{ x_1 a_1 + \dots + x_n a_n : x_1, ..., x_n \in \mathbb{N} \}.$$

The set of holes of S is  $\overline{S} = \mathbb{N} \setminus S$  and  $|\overline{S}| < \infty$  holds.

The Frobenius number and the genus of S are  $\mathfrak{f}(S) = \max \overline{S}$  and  $\mathfrak{g}(S) = |\overline{S}|$ , respectively.

Given  $t \in S \setminus \{0\}$ , the Apéry set of S with respect to t is  $Ap(S,t) = \{s \in S : s - t \in \overline{S}\}.$ 

▲日▼ ▲母▼ ▲日▼ ▲日▼ ヨー シタク

# Introduction

Apéry sets are main tools to study properties of numerical semigroups.

э.

# Introduction

Apéry sets are main tools to study properties of numerical semigroups. For instance, Selmer 1977 gave

(a) 
$$f(S) = \max(\operatorname{Ap}(S, t)) - t$$
,

(b) 
$$g(S) = \frac{1}{t} (\sum_{s \in Ap(S,t)} s) - \frac{t-1}{2}$$

・ロト ・聞ト ・ヨト ・ヨト

# Introduction

Apéry sets are main tools to study properties of numerical semigroups. For instance, Selmer 1977 gave

(a) 
$$f(S) = \max(\operatorname{Ap}(S, t)) - t$$
,

(b) 
$$g(S) = \frac{1}{t} (\sum_{s \in Ap(S,t)} s) - \frac{t-1}{2}$$

It is well known that

 $\mathrm{Ap}(S,t) = \{\alpha_0,...,\alpha_{t-1}: \ \alpha_k \in [k]_t, 0 \le k < t\}.$ 

▲日▼ ▲母▼ ▲日▼ ▲日▼ ヨー シタク

# Introduction

Apéry sets are main tools to study properties of numerical semigroups. For instance, Selmer 1977 gave

(a) 
$$f(S) = \max(\operatorname{Ap}(S, t)) - t$$
,

(b) 
$$g(S) = \frac{1}{t} (\sum_{s \in Ap(S,t)} s) - \frac{t-1}{2}$$

It is well known that

 $Ap(S,t) = \{\alpha_0, ..., \alpha_{t-1} : \alpha_k \in [k]_t, 0 \le k < t\}.$ 

Ap(S, t) has been used for factoring in S.

▲日▼ ▲母▼ ▲日▼ ▲日▼ ヨー シタク

# Introduction

Taking  $G = \text{Cay}(\mathbb{Z}_{a_n}, \{a_1, ..., a_{n-1}\}, \{a_1, ..., a_{n-1}\})$  and any related MDD  $\mathcal{H}$ ,

$$Ap(\langle a_1, ..., a_n \rangle, a_n) = \{ \delta(i_1, ..., i_{n-1}) : [\![i_1, ..., i_{n-1}]\!] \in \mathcal{H} \}.$$

《曰》 《聞》 《臣》 《臣》

- 20

### Introduction

Taking  $G = \text{Cay}(\mathbb{Z}_{a_n}, \{a_1, ..., a_{n-1}\}, \{a_1, ..., a_{n-1}\})$  and any related MDD  $\mathcal{H}$ ,

$$Ap(\langle a_1, ..., a_n \rangle, a_n) = \{ \delta(i_1, ..., i_{n-1}) : [[i_1, ..., i_{n-1}]] \in \mathcal{H} \}.$$

Thus, numerical semigroups and Cayley digraphs are connected through their related minimum distance diagrams.

▲日▼ ▲母▼ ▲日▼ ▲日▼ ヨー シタク

### Introduction

Given  $m \in S = \langle a_1, ..., a_n \rangle$ , a factorization of m in S is  $(x_1, ..., x_n) \in \mathbb{N}^n$  such that  $x_1a_1 + \cdots + x_na_n = m$ .

 $\mathcal{F}(m,S) = \{(x_1,...,x_n) \in \mathbb{N}^n : a_1x_1 + \dots + x_na_n = m\}$ 

The denumerant of m in S is  $d(m, S) = |\mathcal{F}(m, S)|$ .

▲日▼ ▲母▼ ▲日▼ ▲日▼ ヨー シタク

Minimium path diagram of weighted 2–Cayley digrafs Maximum genus of numerical 3–semigroups

# Introduction

Example:  $m = 87, S = \langle 5, 7, 11 \rangle$ 

Seminari CombGraph, MA–IV Some metric results on weighted 2–Cayley digraphs

《曰》 《聞》 《臣》 《臣》

э.

Minimium path diagram of weighted 2–Cayley digrafs Maximum genus of numerical 3–semigroups

# Introduction

### Example: $m = 87, S = \langle 5, 7, 11 \rangle$

												_							
98	103	108	113	118	123	128	133	138	143	148	153	158	163	168	173	178	183	188	193
91	96	101	106	111	116	121	126	131	136	141	146	151	156	161	166	171	176	181	186
84	89	94	99	104	109	114	119	124	129	134	139	144	149	154	159	164	169	174	17 <i>9</i>
77	82	87	92	97	102	107	112	117	122	127	132	137	142	147	152	157	162	167	172
70	75	80	85	90	95	100	105	110	115	120	125	130	135	140	145	150	155	160	165
63	68	73	78	83	88	93	98	103	108	113	118	123	128	133	138	143	148	153	158 <mark> </mark>
56	61	66	71	76	81	86	91	96	101	106	111	116	121	126	131	136	141	146	151
49	54	59	64	69	74	79	84	89	94	99	104	109	114	119	124	129	134	139	144
42	47	52	57	62	67	172	77	82	87	92	97	102	107	112	117	122	127	132	137
35	40	45	50	55	60	65	70	75	80	85	90	95	100	105	110	115	120	125	130
28	33	38	43	48	53	58	63	68	73	78	83	88	93	98	103	108	113	118	123
21	26	31	36	41	46	51	56	61	66	71	76	81	86	91	96	101	106	111	116
14	19	24	29	34	39	44	49	54	59	64	69	74	79	84	89	94	99	104	109
7	12	17	22	27	32	¦37	42	47	52	57	62	67	72	77	82	87	92	97	102 <mark>1</mark>
0	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95
_																			

Seminari CombGraph, MA–IV Some metric results on weighted 2–Cayley digraphs

・ロト ・聞ト ・ヨト ・ヨト

Minimium path diagram of weighted 2–Cayley digrafs Maximum genus of numerical 3–semigroups

# Introduction

### Example: $m = 87, S = \langle 5, 7, 11 \rangle$

	_	_			_						r	_				_	•		r – –
98	103	108	113	118	123	128	133	138	143	148	153	158	163	168	173	178	183	188	193
91	96	101	106	111	116	121	126	131	136	141	146	151	156	161	166	171	176	181	186
84	89	94	99	104	109	114	119	124	129	134	139	144	149	154	159	164	169	174	179
77	82	87	92	97	102	107	112	117	122	127	132	137	142	147	152	157	162	167	172
70	75	80	85	90	95	100	105	110	115	120	125	130	135	140	145	150	155	160	165
63	68	73	78	83	88	93	98	103	108	113	118	123	128	133	138	143	148	153	158
56	61	66	71	76	81	86	91	96	101	106	111	116	121	126	131	136	141	146	151
49	54	59	64	69	74	79	84	89	94	99	104	109	114	119	124	129	134	139	144
42	47	52	57	62	67	172	77	82	87	92	97	102	107	112	117	122	127	132	137
35	40	45	50	55	60	65	70	75	80	85	90	95	100	105	110	115	120	125	130
28	33	38	43	48	53	58	63	68	73	78	83	88	93	98	103	108	113	118	123
21	26	31	36	41	46	51	56	61	66	71	76	81	86	91	96	101	106	111	116
14	19	24	29	34	39	44	49	54	59	64	69	74	79	84	89	94	99	104	109
7	12	17	22	27	32	37	42	47	52	57	62	67	72	77	82	87	92	97	102
0	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95

# $\{(2,0,7)\}$

Seminari CombGraph, MA–IV Some metric results on weighted 2–Cayley digraphs

《曰》 《聞》 《臣》 《臣》

Minimium path diagram of weighted 2-Cayley digrafs

# Introduction

### Example: m = 87, S = (5, 7, 11)

	_	_			_						r	_					+		
98	103	108	113	118	123	128	133	138	143	148	153	158	163	168	173	178	183	188	193
91	96	101	106	111	116	121	126	131	136	141	146	151	156	161	166	171	176	181	186
84	89	94	99	104	109	114	119	124	129	134	139	144	149	154	159	164	169	174	179
77	82	87	92	97	102	107	112	117	122	127	132	137	142	147	152	157	162	167	172
70	75	80	85	90	95	100	105	110	115	120	125	130	135	140	145	150	155	160	165
63	68	73	78	83	88	93	98	103	108	113	118	123	128	133	138	143	148	153	158
56	61	66	71	76	81	86	91	96	101	106	111	116	121	126	131	136	141	146	151
49	54	59	64	69	74	79	84	89	94	99	104	109	114	119	124	129	134	139	144
42	47	52	57	62	67	172	77	82	87	92	97	102	107	112	117	122	127	132	137
35	40	45	50	55	60	65	70	75	80	85	90	95	100	105	110	115	120	125	130
28	33	38	43	48	53	58	63	68	73	78	83	88	93	98	103	108	113	118	123
21	26	31	36	41	46	51	56	61	66	71	76	81	86	91	96	101	106	111	116
14	19	24	29	34	39	44	49	54	59	64	69	74	79	84	89	94	99	104	109
7	12	17	22	27	32	37	42	47	52	57	62	67	72	77	82	87	92	97	102
0	5	$\overline{\mathbb{O}}$	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95

# $\{(2,0,7), (0,3,6)\}$

Seminari CombGraph, MA-IV

Some metric results on weighted 2-Cayley digraphs

Minimium path diagram of weighted 2-Cayley digrafs

# Introduction

### Example: m = 87, S = (5, 7, 11)

98 103 108 113 118 123 128 133 138 143 148 153 158 163 168 173 178 183 188 191 96 101 106 111 116 121 126 131 136 141 146 151 156 161 166 17 178 178 183 188 191 96 191 106 111 1176 121 124 129 134 139 144 149 154 159 156 166 176 177 178 183 188 191 491 49 154 159 156 156 156 156 156 156 156 156 156 156	
91 96 101 106 111 116 121 126 131 136 141 146 151 156 161 166 17 171 76 181 84 89 94 99 104 109 114 119 124 129 134 139 144 149 154 159 164 169 174 77 82 87 92 97 102 107 114 119 124 129 134 139 144 149 154 159 164 169 174 70 75 80 85 90 95 100 105 110 115 120 125 130 135 140 145 150 155 160 63 68 73 78 83 88 93 98 103 108 113 118 123 128 133 140 145 150 155 160 63 68 73 78 83 88 93 98 103 108 113 118 123 128 133 136 143 143 148 153 56 61 66 71 76 81 86 91 96 100 106 1111 16 121 126 131 136 141 146 49 60 59 64 69 74 79 84 89 94 99 104 109 114 119 124 129 134 139	93
84         89         94         99         104         109         114         119         124         129         134         139         144         149         154         159         166         166         174           77         82         87         92         97         102         107         112         117         122         127         132         137         142         147         152         157         166         166         174         154         159         156         166         155         160         155         100         155         100         155         100         155         100         155         100         155         100         155         100         155         100         155         166         135         143         143         143         143         143         143         143         143         143         144         154         156         156         61         66         71         76         81         86         91         96         106         11111         124         124         124         124         124         124         124         129         104         109	86
77 82 87 92 97 102107112117122127132137142147152157162167 70 75 80 85 90 95 100105110115120125130 135140145 150155160 63 68 73 78 83 88 93 98 103108113118123128133140145 150155160 56 61 66 71 76 81 86 91 96 10010611111612112613136141145 49 60 59 64 69 74 79 84 89 94 99 104109114119124129134149 42 47 52 57 62 67 72 77 82 87 92 97 102107112117122127133	79
70       75       80       85       90       95       100105       110115120125130       135140145       150155160         63       68       73       78       83       88       93       98       103108       113118       123128       133138       143148       1531         56       61       66       71       76       81       86       91       96       101106       111116       121126       131136       124124       136141       1461         49       63       59       64       69       74       79       84       89       94       99       104       109       114       124       124       124       124       124       124       127       134       139         42       47       52       57       62       67       72       77       82       87       92       97       102107112       117122127       133	72
63       68       73       78       83       88       93       98       103/108/113/118/123/128/133/138       143/148/153/158         56       61       66       71       76       81       86       91       96       101/106/111116       1211261131136       136/141       146/146         49       59       64       69       74       79       84       89       94       99       104/109/114/119       124/129/134       139/144         42       47       52       57       62       67       72       77       82       87       92       97       102/107/112       117/122/127       132/17	65
56         61         66         71         76         81         86         91         96         101106111111612112613113614114161           49         60         59         64         69         74         79         84         89         94         99         1041091141191241291341391           42         47         52         57         62         67         72         77         82         87         92         97         102107112117122127         1321	58 <mark> </mark>
49 50 59 64 69 74 79 84 89 94 99 104 109 114 119 124 129 134 139 3 42 47 52 57 62 67 72 77 82 87 92 97 102 107 112 117 122 127 132	51
42 47 52 57 62 67 72 77 82 87 92 97 102107112117122127 132	44
	37
35 40 45 50 55 60 65 70 75 80 85 90 95 100 105 110 115 120 125	30
28 33 38 43 48 53 58 63 68 73 78 83 88 93 98 103108 1131181	23
21 26 31 36 41 46 51 56 61 66 71 76 81 86 91 96 101 106 111	16
14 19 24 29 34 39 44 49 54 59 64 69 74 79 84 89 94 99 1041	09
7   12   17 22   27   22   37   42 47   52   57 62   67   72 77   82   87   92   97 1	02
0   5 10 15 20 25 30 35 40 45 50 55 60 65 70 75 80 85 90	95 I

 $\{(2,0,7), (0,3,6), (5,1,5), (3,4,4), (1,7,3)\}$ 

Seminari CombGraph, MA-IV

Some metric results on weighted 2-Cayley digraphs

э

Minimium path diagram of weighted 2–Cayley digrafs Maximum genus of numerical 3–semigroups

### Introduction

### Example: $m = 87, S = \langle 5, 7, 11 \rangle$



# $\{(2,0,7), (0,3,6), (5,1,5), (3,4,4), (1,7,3), (8,2,3), (13,0,2), \\ (6,5,2), (4,8,1), (2,11,0) \}$

Seminari CombGraph, MA–IV Some metric results on weighted 2–Cayley digraphs

Minimium path diagram of weighted 2–Cayley digrafs Maximum genus of numerical 3–semigroups

### Introduction

### Example: $m = 87, S = \langle 5, 7, 11 \rangle$



 $\{ (2,0,7), (0,3,6), (5,1,5), (3,4,4), (1,7,3), (8,2,3), (13,0,2), \\ (6,5,2), (4,8,1), (2,11,0), (11,3,1), (16,1,0), (9,6,0) \}$ 

Seminari CombGraph, MA-IV

Some metric results on weighted 2-Cayley digraphs

### MINIMIUM PATH DIAGRAM OF WEIGHTED 2–CAYLEY DIGRAFS

Seminari CombGraph, MA–IV Some metric results on weighted 2–Cayley digraphs

《曰》 《聞》 《臣》 《臣》

э.

MPD characterization Minimum path enumeration

# Minimum path diagram: definition

We are interested in computing the number of minimum paths between any pair of vertices of  $G = \text{Cay}(G_N, \{a, b\}, \{W_a, W_b\})$ .

MPD characterization Minimum path enumeration

# Minimum path diagram: definition

We are interested in computing the number of minimum paths between any pair of vertices of  $G = \text{Cay}(G_N, \{a, b\}, \{W_a, W_b\})$ .

To this end, we define the minimum path diagram (MPD) associated with G as the unit-square-made region in  $\mathbb{R}_{\geq 0}^2$ ,  $\mathcal{P}(\mathcal{G}_N, \{a, b\}, \{W_a, W_b\})$ , such that it contains each minimum path in G exactly once.

MPD characterization Minimum path enumeration

# Minimum path diagram: definition

We are interested in computing the number of minimum paths between any pair of vertices of  $G = \text{Cay}(G_N, \{a, b\}, \{W_a, W_b\})$ .

To this end, we define the minimum path diagram (MPD) associated with G as the unit-square-made region in  $\mathbb{R}_{\geq 0}^2$ ,  $\mathcal{P}(\mathcal{G}_N, \{a, b\}, \{W_a, W_b\})$ , such that it contains each minimum path in G exactly once.

We need a practical description of  $\mathcal{P}$  to work with. This description will be given from any MDD  $\mathcal{H}$ .

(日) (周) (日) (日) (日)

MPD characterization Minimum path enumeration

# Minimum path diagram: definition

Let us assume  $\mathcal{H}$  is an MDD related to G.

$$q = [l - w - 1, h - 1]$$

$$p = [l - 1, h - y - 1]$$

$$\mathcal{H} = L(l, h, w, y)$$

$$[0, 0]$$

<ロ> (四) (四) (三) (三) (三) (三)

MPD characterization Minimum path enumeration

# Minimum path diagram: definition

Let us assume  $\mathcal{H}$  is an MDD related to G.

$$q = [l - w - 1, h - 1]$$

$$p = [l - 1, h - y - 1]$$

$$\mathcal{H} = L(l, h, w, y)$$

$$[0, 0]$$

Inequality  $(lW_a - yW_b)(hW_b - wW_a) \ge 0$  holds and both factors don't vanish.

Seminari CombGraph, MA–IV Some metric results on weighted 2–Cayley digraphs

MPD characterization Minimum path enumeration

# Minimum path diagram: definition

Let us assume  $\mathcal{H}$  is an MDD related to G.

$$q = [l - w - 1, h - 1]$$

$$p = [l - 1, h - y - 1]$$

$$\mathcal{H} = L(l, h, w, y)$$

$$[0, 0]$$

Inequality  $(lW_a - yW_b)(hW_b - wW_a) \ge 0$  holds and both factors don't vanish.

Set  $\boldsymbol{u} = (l, -y)$  and  $\boldsymbol{v} = (-w, h)$ .

イロト イロト イヨト イヨト 三国

MPD characterization Minimum path enumeration

# Minimum path diagram: characterization

Theorem 1 (characterization of  $\mathcal{P}$ )

・ロト ・聞ト ・ヨト ・ヨト

MPD characterization Minimum path enumeration

### Minimum path diagram: characterization

Theorem 1 (characterization of  $\mathcal{P}$ ) (a) If either  $\delta(p) = \delta(q)$ , or  $\delta(p) < \delta(q)$  and  $lW_a > yW_b$ , or  $\delta(p) > \delta(q)$  and  $hW_b > wW_a$ , then  $\mathcal{P}(\mathbf{G}_N, \{a, b\}, \{W_a, W_b\}) = \mathcal{H}.$ 

< ロト ( 母 ) ( ヨ ) ( コ ) ( コ ) ( 1 )

MPD characterization Minimum path enumeration

### Minimum path diagram: characterization

Theorem 1 (characterization of  $\mathcal{P}$ ) (a) If either  $\delta(p) = \delta(q)$ , or  $\delta(p) < \delta(q)$  and  $lW_a > yW_b$ , or  $\delta(p) > \delta(q)$  and  $hW_b > wW_a$ , then  $\mathcal{P}(\mathbf{G}_N, \{a, b\}, \{W_a, W_b\}) = \mathcal{H}.$ 

(b) If  $\delta(p) < \delta(q)$  and  $lW_a = yW_b$ , then  $\mathcal{P}(\mathcal{G}_N, \{a, b\}, \{W_a, W_b\}) = \bigcup_{\lambda=0}^{\lfloor \frac{h-1}{y} \rfloor} \Delta(q + \lambda \boldsymbol{u}).$ 

< ロト ( 母 ) ( ヨ ) ( コ ) ( コ ) ( 1 )

MPD characterization Minimum path enumeration

### Minimum path diagram: characterization

Theorem 1 (characterization of  $\mathcal{P}$ ) (a) If either  $\delta(p) = \delta(q)$ , or  $\delta(p) < \delta(q)$  and  $lW_a > yW_b$ , or  $\delta(p) > \delta(q)$  and  $hW_b > wW_a$ , then  $\mathcal{P}(\mathbf{G}_N, \{a, b\}, \{W_a, W_b\}) = \mathcal{H}.$ 

(b) If  $\delta(p) < \delta(q)$  and  $lW_a = yW_b$ , then  $\mathcal{P}(\mathcal{G}_N, \{a, b\}, \{W_a, W_b\}) = \bigcup_{\lambda=0}^{\lfloor \frac{h-1}{y} \rfloor} \Delta(q + \lambda \boldsymbol{u}).$ 

(c) If  $\delta(p) > \delta(q)$  and  $hW_b = wW_a$ , then  $\lfloor \frac{l-1}{w} \rfloor$ 

$$\mathcal{P}(\mathbf{G}_N, \{a, b\}, \{W_a, W_b\}) = \bigcup_{\substack{\lambda = 0 \\ \langle \Box \rangle \land \langle \Box \rangle \land \langle \Box \rangle \land \langle \Box \rangle \land \langle \Xi \rangle \: \langle \Xi \rangle \land \langle \Xi \rangle \: \langle \Xi \rangle \: \langle \Xi \rangle \land \langle \Xi \rangle \land \langle \Xi \rangle \land \langle \Xi \rangle \land \langle \Xi \rangle \: \langle \Xi \land \langle \Xi \land \langle \Xi \rangle \: \langle \Box \land \langle \Xi \land \langle \Xi \land \langle \Xi \rangle \: \langle \Xi \land \langle \Xi \land \langle \Xi \land \Box \land \langle \Xi \land \Box \land \langle \Xi \land \langle \Xi \land \Box \: \langle \Xi \land \Box \land \langle \Xi \land \Box \: \langle \Xi \land \langle \Xi$$

Seminari CombGraph, MA–IV Some metric results on weighted 2–Cayley digraphs

MPD characterization Minimum path enumeration

### Minimum path diagram: characterization

Example 4:  $G_1 = \text{Cay}(\mathbb{Z}_3 \times \mathbb{Z}_3, \{(1,0), (0,1)\}, \{1,1\})$ 

Seminari CombGraph, MA–IV Some metric results on weighted 2–Cayley digraphs
MPD characterization Minimum path enumeration

#### Minimum path diagram: characterization

Example 4:  $G_1 = \text{Cay}(\mathbb{Z}_3 \times \mathbb{Z}_3, \{(1,0), (0,1)\}, \{1,1\})$ , with DDM  $\mathcal{H} = L(3,3,0,0)$ .

(0,2)	(1,2)	(2,2)
(0,1)	(1,1)	(2,1)
(0,0)	(1,0)	(2,0)

MPD characterization Minimum path enumeration

#### Minimum path diagram: characterization

Example 4:  $G_1 = \text{Cay}(\mathbb{Z}_3 \times \mathbb{Z}_3, \{(1,0), (0,1)\}, \{1,1\})$ , with DDM  $\mathcal{H} = L(3,3,0,0)$ . Then,  $p = q = \llbracket 2,2 \rrbracket$ 

(0,2)	(1,2)	(2,2)
(0,1)	(1,1)	(2,1)
(0,0)	(1,0)	(2,0)

MPD characterization Minimum path enumeration

#### Minimum path diagram: characterization

Example 4:  $G_1 = \text{Cay}(\mathbb{Z}_3 \times \mathbb{Z}_3, \{(1,0), (0,1)\}, \{1,1\})$ , with DDM  $\mathcal{H} = L(3,3,0,0)$ . Then,  $p = q = [\![2,2]\!]$  and  $\delta(p) = \delta(q) = 4$ ,

(0,2)	(1,2)	(2,2)
(0,1)	(1,1)	(2,1)
(0,0)	(1,0)	(2,0)

MPD characterization Minimum path enumeration

#### Minimum path diagram: characterization

Example 4:  $G_1 = \text{Cay}(\mathbb{Z}_3 \times \mathbb{Z}_3, \{(1,0), (0,1)\}, \{1,1\})$ , with DDM  $\mathcal{H} = L(3,3,0,0)$ . Then,  $p = q = [\![2,2]\!]$  and  $\delta(p) = \delta(q) = 4$ , so  $\mathcal{P} = \mathcal{H}$ .

(0,2)	(1,2)	(2,2)
(0,1)	(1,1)	(2,1)
(0,0)	(1,0)	(2,0)

MPD characterization Minimum path enumeration

#### Minimum path diagram: characterization

Example 5:  $G_2 = \operatorname{Cay}(\mathbb{Z}_{12}, \{1, 2\}, \{1, 2\})$ 

Seminari CombGraph, MA–IV Some metric results on weighted 2–Cayley digraphs

MPD characterization Minimum path enumeration

#### Minimum path diagram: characterization

Example 5:  $G_2 = \text{Cay}(\mathbb{Z}_{12}, \{1, 2\}, \{1, 2\})$ , with DDM  $\mathcal{H} = \text{L}(12, 1, 2, 0)$ . Then,  $p = \llbracket 11, 0 \rrbracket \sim 11$  and  $q = \llbracket 9, 0 \rrbracket \sim 9$ . Thus,  $\delta(p) = 11 > \delta(q) = 9$  and  $hW_b = 2 = wW_a$ .

MPD characterization Minimum path enumeration

#### Minimum path diagram: characterization

Example 5:  $G_2 = \text{Cay}(\mathbb{Z}_{12}, \{1, 2\}, \{1, 2\})$ , with DDM  $\mathcal{H} = \text{L}(12, 1, 2, 0)$ . Then,  $p = \llbracket 11, 0 \rrbracket \sim 11$  and  $q = \llbracket 9, 0 \rrbracket \sim 9$ . Thus,  $\delta(p) = 11 > \delta(q) = 9$  and  $hW_b = 2 = wW_a$ . Item (c) of Theorem 1 holds and  $\mathcal{P} = \bigcup_{\lambda=0}^5 \Delta((11, 0) + \lambda(-2, 4))$ 

Minimium path diagram of weighted 2-Cavley digrafs

MPD characterization

#### Minimum path diagram: characterization

Example 5:  $G_2 = \text{Cay}(\mathbb{Z}_{12}, \{1, 2\}, \{1, 2\})$ , with DDM  $\mathcal{H} = L(12, 1, 2, 0)$ . Then,  $p = [11, 0] \sim 11$  and  $q = [9, 0] \sim 9$ . Thus,  $\delta(p) = 11 > \delta(q) = 9$  and  $hW_b = 2 = wW_a$ . Item (c) of Theorem 1 holds and  $\mathcal{P} = \bigcup_{\lambda=0}^{5} \Delta((11,0) + \lambda(-2,4))$ 



Some metric results on weighted 2-Cayley digraphs

(日) (四) (日) (日) (日)

Seminari CombGraph, MA-IV

MPD characterization Minimum path enumeration

## Minimum path diagram: enumeration

The MPD characterization allow us to enumerate the minimum paths from 0 in G.

《曰》 《聞》 《臣》 《臣》

- 20

MPD characterization Minimum path enumeration

# Minimum path diagram: enumeration

The MPD characterization allow us to enumerate the minimum paths from 0 in G.

Let us assume that we can efficiently compute the unit square associated with a given vertex u in  $\mathcal{H}$ , i.e.  $u \sim [\![i_0, j_0]\!] \in \mathcal{H}$  (A. and Barguilla, 2008).

MPD characterization Minimum path enumeration

# Minimum path diagram: enumeration

Theorem 2 (enumeration)

・ロト ・聞ト ・ヨト ・ヨト

MPD characterization Minimum path enumeration

## Minimum path diagram: enumeration

Theorem 2 (enumeration)

Given  $g \sim [\![i_0, j_0]\!] \in \mathcal{H} = L(l, h, w, y)$ , where  $\mathcal{H}$  is an MDD associated with G, let  $\mathcal{N}(g, G)$  be the number of minimum paths from 0 to g in G.

MPD characterization Minimum path enumeration

## Minimum path diagram: enumeration

Theorem 2 (enumeration)

Given  $g \sim \llbracket i_0, j_0 \rrbracket \in \mathcal{H} = L(l, h, w, y)$ , where  $\mathcal{H}$  is an MDD associated with G, let  $\mathcal{N}(g, G)$  be the number of minimum paths from 0 to g in G. Then,

(a) if Theorem 1–(a) holds, then  $\mathcal{N}(g,G) = \binom{i_0+j_0}{i_0}$ ,

MPD characterization Minimum path enumeration

## Minimum path diagram: enumeration

Theorem 2 (enumeration)

Given  $g \sim [\![i_0, j_0]\!] \in \mathcal{H} = L(l, h, w, y)$ , where  $\mathcal{H}$  is an MDD associated with G, let  $\mathcal{N}(g, G)$  be the number of minimum paths from 0 to g in G. Then,

(a) if Theorem 1–(a) holds, then  $\mathcal{N}(g,G) = {i_0+j_0 \choose i_0}$ ,

(b) if Theorem 1–(b) holds, then

$$\mathcal{N}(g,G) = \sum_{\lambda=0}^{\lfloor \frac{j_0}{y} \rfloor} \binom{i_0 + j_0 + \lambda(l-y)}{j_0 - \lambda y},$$

Minimium path diagram of weighted 2-Cavley digrafs

MPD characterization Minimum path enumeration

# Minimum path diagram: enumeration

Theorem 2 (enumeration)

Given  $q \sim [i_0, j_0] \in \mathcal{H} = L(l, h, w, y)$ , where  $\mathcal{H}$  is an MDD associated with G, let  $\mathcal{N}(q,G)$  be the number of minimum paths from 0 to q in G. Then,

- (a) if Theorem 1–(a) holds, then  $\mathcal{N}(g,G) = \begin{pmatrix} i_0+j_0 \\ i_0 \end{pmatrix}$ ,
- (b) if Theorem 1–(b) holds, then

$$\mathcal{N}(g,G) = \sum_{\lambda=0}^{\lfloor \frac{j_0}{y} \rfloor} \binom{i_0 + j_0 + \lambda(l-y)}{j_0 - \lambda y},$$

(c) if Teorema 1-(c) holds, then

$$\mathcal{N}(g,G) = \sum_{\lambda=0}^{\lfloor \frac{i_0}{w} \rfloor} \binom{i_0 + j_0 + \lambda(h-w)}{i_0 - \lambda w}.$$

MPD characterization Minimum path enumeration

#### Minimum path diagram: enumeration

Example 6: Minimum paths from (0,0) to (2,2) in  $G_3 = \text{Cay}(\mathbb{Z}_3 \times \mathbb{Z}_3, \{(1,0), (0,1)\}, \{1,1\})$ 

Seminari CombGraph, MA–IV Some metric results on weighted 2–Cayley digraphs

MPD characterization Minimum path enumeration

#### Minimum path diagram: enumeration

Example 6: Minimum paths from (0,0) to (2,2) in  $G_3 = \text{Cay}(\mathbb{Z}_3 \times \mathbb{Z}_3, \{(1,0), (0,1)\}, \{1,1\})$  are



Seminari CombGraph, MA–IV

Some metric results on weighted 2–Cayley digraphs

MPD characterization Minimum path enumeration

## Minimum path diagram: enumeration

#### Example 6: Theorem 2–(a) holds with $(2,2) \sim [\![2,2]\!]$ .

Seminari CombGraph, MA–IV Some metric results on weighted 2–Cayley digraphs

MPD characterization Minimum path enumeration

## Minimum path diagram: enumeration

Example 6: Theorem 2–(a) holds with  $(2, 2) \sim [2, 2]$ . Then,

$$\mathcal{N}((2,2),G_3) = \binom{2+2}{2} = 6.$$

Seminari CombGraph, MA–IV Some metric results on weighted 2–Cayley digraphs

MPD characterization Minimum path enumeration

# Minimum path diagram: enumeration

Example 7:  $G_4 = \text{Cay}(\mathbb{Z}_{500}, \{1, 2\}, \{1, 1\})$ 

(日) (四) (日) (日) (日)

MPD characterization Minimum path enumeration

## Minimum path diagram: enumeration

Example 7:  $G_4 = \text{Cay}(\mathbb{Z}_{500}, \{1, 2\}, \{1, 1\})$ 

 $G_4$  has related the MDD L(2, 250, 0, 1)

MPD characterization Minimum path enumeration

## Minimum path diagram: enumeration

Example 7:  $G_4 = \text{Cay}(\mathbb{Z}_{500}, \{1, 2\}, \{1, 1\})$ 

 $G_4$  has related the MDD L(2, 250, 0, 1) and

#MP to p = 497: 249, #MP to q = 499: 250.

MPD characterization Minimum path enumeration

## Minimum path diagram: enumeration

Example 7:  $G_5 = \text{Cay}(\mathbb{Z}_{500}, \{1, 2\}, \{1, 2\})$ 

Seminari CombGraph, MA–IV Some metric results on weighted 2–Cayley digraphs

MPD characterization Minimum path enumeration

### Minimum path diagram: enumeration

Example 7:  $G_5 = \text{Cay}(\mathbb{Z}_{500}, \{1, 2\}, \{1, 2\})$ 

 $G_5$  has related the MDDs L(2, 250, 0, 1) and L(500, 1, 2, 0)

# Minimum path diagram: enumeration

Example 7:  $G_5 = \text{Cay}(\mathbb{Z}_{500}, \{1, 2\}, \{1, 2\})$ 

 $G_5$  has related the MDDs  $\mathcal{L}(2,250,0,1)$  and  $\mathcal{L}(500,1,2,0)$  and

 $\label{eq:mp_stable} \begin{array}{l} \# \mathrm{MP} \mbox{ to } p = 497; \\ 1394232245616978801397243828704072839500702565876973072641089 \\ 62948325571622863290691557658876222521294125, \\ \# \mathrm{MP} \mbox{ to } q = 499; \\ 5325493296145942940693607070474249585412918826163642393957905 \\ 9478176515507039697978099330699648074089624. \end{array}$ 

Introduction	General comments
Minimium path diagram of weighted 2–Cayley digrafs	Problem statement
Maximum genus of numerical 3–semigroups	Solution

#### MAXIMUM GENUS OF NUMERICAL 3-SEMIGROUPS

Seminari CombGraph, MA–IV Some metric results on weighted 2–Cayley digraphs

《曰》 《聞》 《臣》 《臣》

General comments Problem statement Solution

# Genus: general comments

Closed expressions for  $\mathfrak{f}(S)$  and  $\mathfrak{g}(S)$  are known for 2–semigroups only,

$$\begin{aligned} \mathfrak{f}(\langle a,b\rangle) &= (a-1)(b-1)-1, & \text{Frobenius ?} \\ \mathfrak{g}(\langle a,b\rangle) &= \frac{1}{2}(a-1)(b-1). & \text{Sylvester 1884} \end{aligned}$$

(日) (圖) (필) (필) (필)

General comments Problem statement Solution

# Genus: general comments

Closed expressions for  $\mathfrak{f}(S)$  and  $\mathfrak{g}(S)$  are known for 2-semigroups only,

$$\begin{aligned} \mathfrak{f}(\langle a,b\rangle) &= (a-1)(b-1)-1, & \text{Frobenius ?} \\ \mathfrak{g}(\langle a,b\rangle) &= \frac{1}{2}(a-1)(b-1). & \text{Sylvester 1884} \end{aligned}$$

Many authors paid attention to find closed expressions for f(S) before 1990.

< ロト ( 母 ) ( ヨ ) ( コ ) ( コ ) ( コ ) ( コ ) ( コ ) ( コ ) ( コ ) ( コ ) ( 1 )

General comments Problem statement Solution

# Genus: general comments

Closed expressions for  $\mathfrak{f}(S)$  and  $\mathfrak{g}(S)$  are known for 2–semigroups only,

$$\begin{aligned} \mathfrak{f}(\langle a,b\rangle) &= (a-1)(b-1)-1, & \text{Frobenius ?} \\ \mathfrak{g}(\langle a,b\rangle) &= \frac{1}{2}(a-1)(b-1). & \text{Sylvester 1884} \end{aligned}$$

Many authors paid attention to find closed expressions for  $\mathfrak{f}(S)$  before 1990.

Curtis 1990 proved the non-existence of polynomial expression of f(S) for 3-semigroups.

< ロト ( 母 ) ( ヨ ) ( コ ) ( コ ) ( コ ) ( コ ) ( コ ) ( コ ) ( コ ) ( コ ) ( 1 )

General comments Problem statement Solution

# Genus: general comments

Lewin 1972 gave the following sharp upper bound of  $\mathfrak{f}(S)$  for 3–semigroups

$$\mathbf{F}(N) = \max_{\substack{1 < a < b < c \le N \\ \gcd(a,b,c) = 1}} \mathfrak{f}(\langle a, b, c \rangle) = \begin{cases} \frac{1}{2}(N-2)^2 - 1, & N \text{ even,} \\ \\ \frac{1}{2}(N-3)(N-1) - 1, & N \text{ odd.} \end{cases}$$

General comments Problem statement Solution

# Genus: general comments

Lewin 1972 gave the following sharp upper bound of  $\mathfrak{f}(S)$  for 3–semigroups

$$\mathbf{F}(N) = \max_{\substack{1 < a < b < c \le N \\ \gcd(a,b,c) = 1}} \mathfrak{f}(\langle a, b, c \rangle) = \begin{cases} \frac{1}{2}(N-2)^2 - 1, & N \text{ even,} \\ \\ \frac{1}{2}(N-3)(N-1) - 1, & N \text{ odd.} \end{cases}$$

Erdős and Graham 1972 conjectured that critical semigroups are  $S_1 = \langle N-2, N-1, N \rangle$  and  $S_2 = \langle N/2, N-1, N \rangle$  for even  $N \ge 4$ , and  $S_3 = \langle (N-1)/2, N-1, N \rangle$  for odd  $N \ge 5$ .

< ロト ( 母 ) ( ヨ ) ( コ ) ( コ ) ( コ ) ( コ ) ( コ ) ( コ ) ( コ ) ( コ ) ( 1 )

General comments Problem statement Solution

# Genus: general comments

Lewin 1972 gave the following sharp upper bound of  $\mathfrak{f}(S)$  for 3–semigroups

$$\mathbf{F}(N) = \max_{\substack{1 < a < b < c \le N \\ \gcd(a,b,c) = 1}} \mathfrak{f}(\langle a, b, c \rangle) = \begin{cases} \frac{1}{2}(N-2)^2 - 1, & N \text{ even,} \\ \\ \frac{1}{2}(N-3)(N-1) - 1, & N \text{ odd.} \end{cases}$$

Erdős and Graham 1972 conjectured that critical semigroups are  $S_1 = \langle N-2, N-1, N \rangle$  and  $S_2 = \langle N/2, N-1, N \rangle$  for even  $N \ge 4$ , and  $S_3 = \langle (N-1)/2, N-1, N \rangle$  for odd  $N \ge 5$ .

Dixmier 1990 proved this conjecture.

< ロト ( 母 ) ( ヨ ) ( コ ) ( コ ) ( コ ) ( コ ) ( コ ) ( コ ) ( コ ) ( コ ) ( 1 )

General comments Problem statement Solution

# Genus: general comments

Lewin 1972 gave the following sharp upper bound of  $\mathfrak{f}(S)$  for 3–semigroups

$$\mathbf{F}(N) = \max_{\substack{1 < a < b < c \le N \\ \gcd(a,b,c) = 1}} \mathfrak{f}(\langle a, b, c \rangle) = \begin{cases} \frac{1}{2}(N-2)^2 - 1, & N \text{ even,} \\ \\ \frac{1}{2}(N-3)(N-1) - 1, & N \text{ odd.} \end{cases}$$

Erdős and Graham 1972 conjectured that critical semigroups are  $S_1 = \langle N-2, N-1, N \rangle$  and  $S_2 = \langle N/2, N-1, N \rangle$  for even  $N \ge 4$ , and  $S_3 = \langle (N-1)/2, N-1, N \rangle$  for odd  $N \ge 5$ .

Dixmier 1990 proved this conjecture.

Hamidoune 1998 gave more general results for  $n \geq 3$ .

General comments Problem statement Solution

Genus: problem statement

What can we say about the function

$$\mathbf{G}(N) = \max_{\substack{1 < a < b < c \le N, \\ \gcd(a, b, c) = 1}} \mathfrak{g}(\langle a, b, c \rangle).$$

Seminari CombGraph, MA–IV Some metric results on weighted 2–Cayley digraphs

- 20

General comments Problem statement Solution

# Genus: problem statement

What can we say about the function

$$\mathbf{G}(N) = \max_{\substack{1 < a < b < c \le N, \\ \gcd(a,b,c) = 1}} \mathfrak{g}(\langle a,b,c \rangle).$$

Not known results.

General comments Problem statement Solution

# Genus: problem statement

c	Max g	Critical semigrups	c	Max g	Critical semigrups
4	1	$\langle 2,3,4  angle$	13	30	$\langle 6, 12, 13 \rangle, \langle 11, 12, 13 \rangle$
5	2	$\langle 2,4,5\rangle,\langle 3,4,5\rangle$	14	36	$\langle 7, 13, 14 \rangle, \langle 12, 13, 14 \rangle$
6	4	$\langle 3,5,6 angle,\langle 4,5,6 angle$	15	42	$\langle 7, 14, 15 \rangle, \langle 13, 14, 15 \rangle$
7	6	$\langle 3,6,7 angle,\langle 5,6,7 angle$	16	49	$\langle 8, 15, 16 \rangle, \langle 14, 15, 16 \rangle$
8	9	$\langle 4,7,8 angle, \langle 6,7,8 angle$	17	56	$\langle 8, 16, 17 \rangle, \langle 15, 16, 17 \rangle$
9	12	$\langle 4, 8, 9 \rangle, \langle 7, 8, 9 \rangle$	18	64	$\langle 9, 17, 18 \rangle, \langle 16, 17, 18 \rangle$
10	16	$\langle 5,9,10 angle, \langle 8,9,10 angle$	19	72	$\langle 9, 18, 19 \rangle, \langle 17, 18, 19 \rangle$
11	20	$\langle 5, 10, 11 \rangle, \langle 9, 10, 11 \rangle$	20	81	$\langle 10, 19, 20 \rangle, \langle 18, 19, 20 \rangle$
12	25	$\langle 6, 11, 12 \rangle, \langle 10, 11, 12 \rangle$	21	90	$\langle 10, 20, 21 \rangle, \langle 19, 20, 21 \rangle$

Maximum genus and critical semigroups for  $c \in \{4, \ldots, 21\}$ 

Seminari CombGraph, MA–IV Some metric results on weighted 2–Cayley digraphs
General comments Problem statement Solution

## Genus: solution

Example 8: Given  $S = \langle 4, 9, 11 \rangle$ , the digraph  $Cay(\mathbb{Z}_{11}, \{4, 9\}, \{4, 9\})$  has related the MDD L(5, 3, 4, 1).

Seminari CombGraph, MA–IV Some metric results on weighted 2–Cayley digraphs

・ロト ・四ト ・ヨト ・ヨト

э

General comments Problem statement Solution

### Genus: solution

Example 8: Given  $S = \langle 4, 9, 11 \rangle$ , the digraph  $Cay(\mathbb{Z}_{11}, \{4, 9\}, \{4, 9\})$  has related the MDD L(5, 3, 4, 1).



<ロ> (日) (日) (日) (日) (日)

General comments Problem statement Solution

## Genus: solution

Example 8: Given  $S = \langle 4, 9, 11 \rangle$ , the digraph  $Cay(\mathbb{Z}_{11}, \{4, 9\}, \{4, 9\})$  has related the MDD L(5, 3, 4, 1).



 $Ap(\langle 4, 9, 11 \rangle, 11) = \{0, 4, 8, 9, 12, 13, 16, 17, 18, 21, 25\}$ 

Seminari CombGraph, MA–IV Some metric results on weighted 2–Cayley digraphs

General comments Problem statement Solution

## Genus: solution

Example 8: Given  $S = \langle 4, 9, 11 \rangle$ , the digraph  $Cay(\mathbb{Z}_{11}, \{4, 9\}, \{4, 9\})$  has related the MDD L(5, 3, 4, 1).



 $\begin{array}{l} \operatorname{Ap}(\langle 4,9,11\rangle,11)=\{0,4,8,9,12,13,16,17,18,21,25\}\\ \overline{S}=\{1,2,3,5,6,7,10,14\} \end{array}$ 

Seminari CombGraph, MA–IV Some metric results on weighted 2–Cayley digraphs

General comments Problem statement Solution

## Genus: solution

Example 8: Given  $S = \langle 4, 9, 11 \rangle$ , the digraph  $Cay(\mathbb{Z}_{11}, \{4, 9\}, \{4, 9\})$  has related the MDD L(5, 3, 4, 1).



Seminari CombGraph, MA–IV Some metric results on weighted 2–Cayley digraphs

General comments Problem statement Solution

## Genus: solution

Let L(l, h, w, y) be an MDD associated with the semigroup  $S = \langle a, b, c \rangle$ . Then,

$$\mathfrak{g}(S) = \frac{l(h-y)}{2c} [(l-1)a + (h-y-1)b] + \frac{y(l-w)}{2c} [(l-w-1)a + (2h-y-1)b] - \frac{c-1}{2}. \quad (*)$$

Let g(a, b, c, l, h, w, y) be defined by (\*) on the compact K given by the restrictions  $4 \le a + 2 \le b + 1 \le c \le N$ ,  $1 \le w + 1 \le l \le c$ ,  $1 \le y + 1 \le h \le c$  and lh - wy = c.

▲日▼ ▲母▼ ▲日▼ ▲日▼ ヨー シタク

General comments Problem statement Solution

#### Genus: solution

Set  $\boldsymbol{x} = (a, b, c, l, h, w, y) \in K$  and consider

 $K = K_1 \cup K_2 \cup U$ 

with

$$\begin{array}{lll} K_1 &=& \{ {\bm x} \in K: \ wy = 0 \}, \\ K_2 &=& \{ {\bm x} \in K: \ wy \ge 1 \}, \\ U &=& \{ {\bm x} \in K: \ 0 < wy < 1 \}. \end{array}$$

(日) (문) (문) (문) (문)

General comments Problem statement Solution

## Genus: solution

Set  $\boldsymbol{x} = (a, b, c, l, h, w, y) \in K$  and consider

 $K = K_1 \cup K_2 \cup U$ 

with

$$\begin{aligned} K_1 &= \{ \pmb{x} \in K : \ wy = 0 \}, \\ K_2 &= \{ \pmb{x} \in K : \ wy \ge 1 \}, \\ U &= \{ \pmb{x} \in K : \ 0 < wy < 1 \}. \end{aligned}$$

Clearly MDDs given by points in U do not make sense.

General comments Problem statement Solution

## Genus: solution

Set  $\boldsymbol{x} = (a, b, c, l, h, w, y) \in K$  and consider

 $K = K_1 \cup K_2 \cup U$ 

with

$$K_1 = \{ \mathbf{x} \in K : wy = 0 \}, K_2 = \{ \mathbf{x} \in K : wy \ge 1 \}, U = \{ \mathbf{x} \in K : 0 < wy < 1 \}.$$

Clearly MDDs given by points in U do not make sense.

Thus, search can be restricted on the compact  $K_1 \cup K_2$ .

< ロト ( 母 ) ( ヨ ) ( コ ) ( コ ) ( コ ) ( コ ) ( コ ) ( コ ) ( コ ) ( コ ) ( コ ) ( 1 )

General comments Problem statement Solution

# Genus: solution

Criticial points of g are located in the border of  $K_1 \cup K_2$ .

《曰》 《聞》 《臣》 《臣》

General comments Problem statement Solution

# Genus: solution

Criticial points of g are located in the border of  $K_1 \cup K_2$ .  $\partial K_1$ :

(1) For even 
$$N \ge 4$$
:  $g = \frac{1}{4}(N-2)^2$  at  
 $l = 2, h = N/2, w = 1, y = 0, a = N/2, b = N-1, c = N$  and  
 $l = N/2, h = 2, w = 1, y = 0, a = N-2, b = N-1, c = N$ .  
(2) For odd  $N > 5$ :  $g = \frac{1}{4}(N-3)(N-1)$  at

 $l = N, h = 1, w = 2, y = 0, a = \frac{1}{2}(N-1), b = N-1, c = N.$ 

▲日▼ ▲母▼ ▲日▼ ▲日▼ ヨー シタク

General comments Problem statement Solution

# Genus: solution

Criticial points of g are located in the border of  $K_1 \cup K_2$ .  $\partial K_1$ :

(1) For even  $N \ge 4$ :  $g = \frac{1}{4}(N-2)^2$  at l = 2, h = N/2, w = 1, y = 0, a = N/2, b = N-1, c = N and l = N/2, h = 2, w = 1, y = 0, a = N-2, b = N-1, c = N.(2) For odd  $N \ge 5$ :  $g = \frac{1}{4}(N-3)(N-1)$  at

 $l = N, h = 1, w = 2, y = 0, a = \frac{1}{2}(N-1), b = N-1, c = N.$  $\partial K_2$ :

- (3) For even  $N \ge 4$ : no valid point attains  $g = \frac{1}{4}(N-2)^2$ .
- (4) For odd  $N \ge 5$ :  $g = \frac{1}{4}(N-3)(N-1)$  at  $l = \frac{1}{2}(N+1), h = 2, w = 1, y = 1, a = N-2, b = N-1, c = N$ and  $l = 2, h = \frac{1}{2}(N+1), w = 1, y = 1, a = \frac{1}{2}(N-1), b = N-1, c = N.$

General comments Problem statement Solution

#### Genus: solution

#### Theorem

$$\mathcal{G}(N) = \left\{ \begin{array}{ll} \frac{1}{4}(N-2)^2, & N \text{ even}, \\ \\ \frac{1}{4}(N-3)(N-1), & N \text{ odd}. \end{array} \right.$$

Critical semigroups are almost those of F(N):

- (i)  $S_1 = \langle N-2, N-1, N \rangle$  and  $S_2 = \langle N/2, N-1, N \rangle$  for even  $N \ge 4$ .
- (ii)  $S_1$  and  $S_3 = \langle (N-1)/2, N-1, N \rangle$  for odd  $N \ge 5$ .

▲日▼ ▲母▼ ▲日▼ ▲日▼ ヨー シタク