Some metric results on weighted 2–Cayley digraphs

Seminari CombGraph, MA–IV

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The length of a path is the sum of the weighs of his arcs.

A minimum path from u to v is a path of minimum lentgh over all paths from u to v.

The distance from u to v is the lentgh of a minimum path from u to v.

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These diagrams can be seen as a geometric view of the hole digraph. They can assist intuition when working with distance–related ideas and reasonings.

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\begin{aligned}
[\![i_1, \ldots, i_n]\!] &= [i_1, i_1+1] \times \cdots \times [i_n, i_n+1] \in \mathbb{R}^n, \\
\delta(i_1, \ldots, i_n) &= i_1 W_1 + \cdots + i_n W_n, \\
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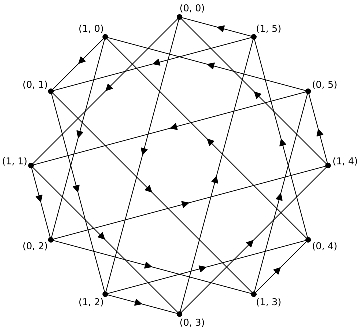

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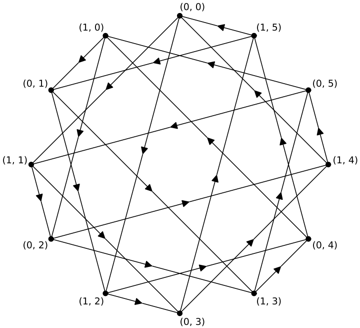

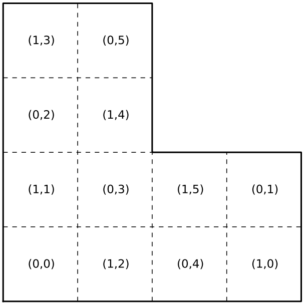

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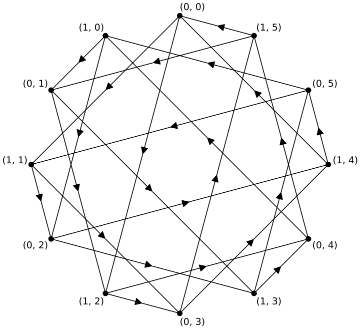

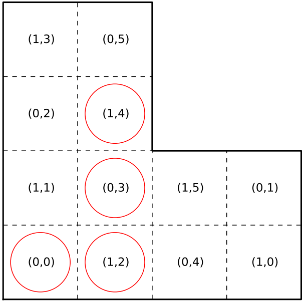

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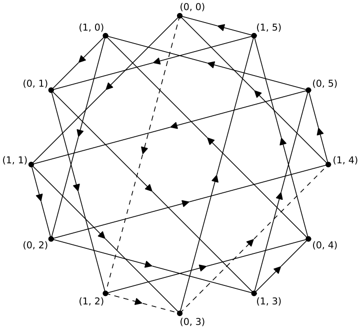

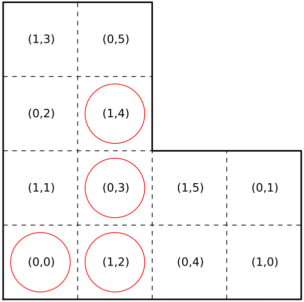

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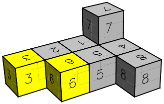

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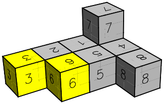

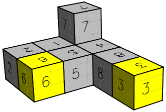

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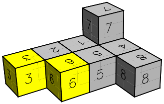

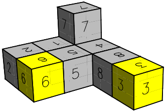

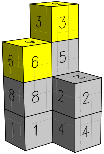

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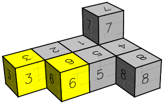

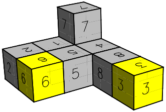

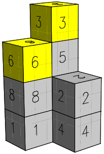

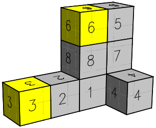

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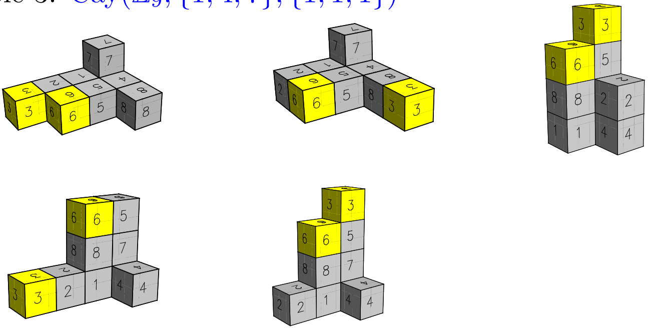

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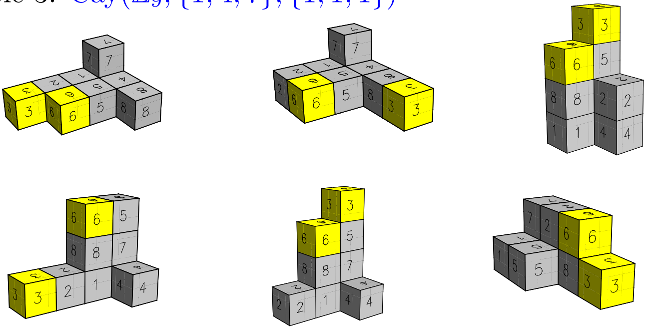

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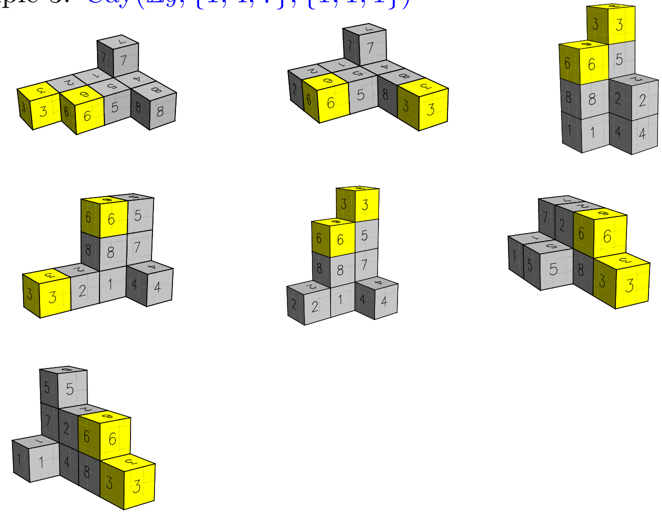

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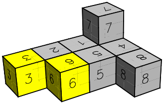
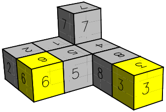
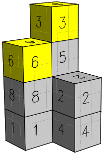
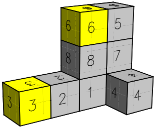
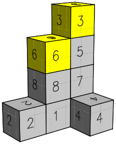
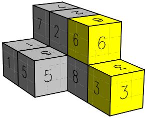
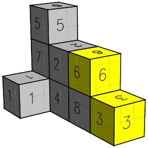
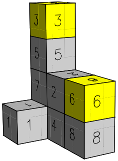

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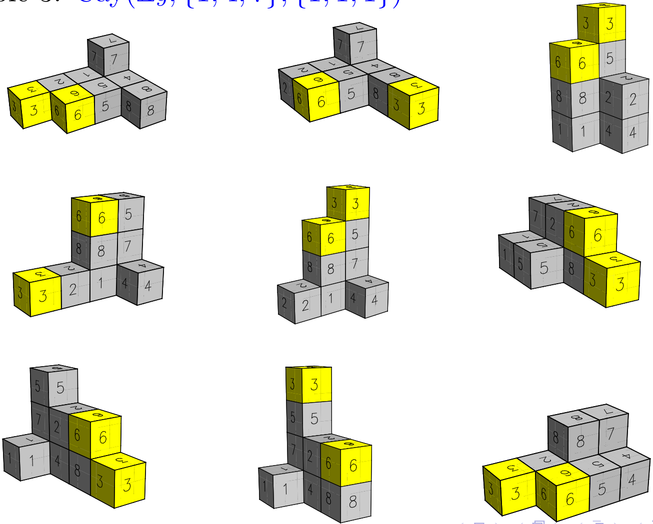

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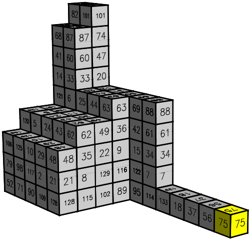

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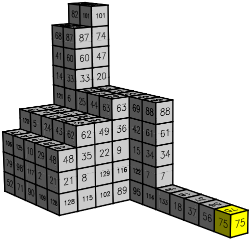

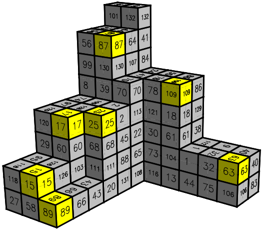

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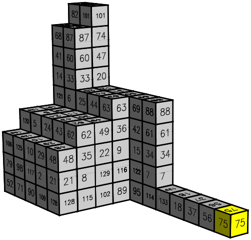

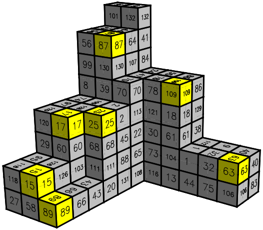

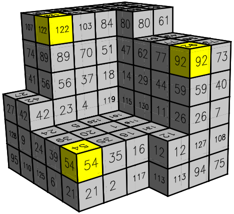

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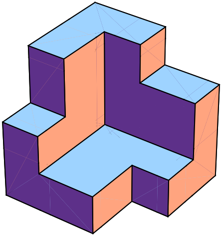

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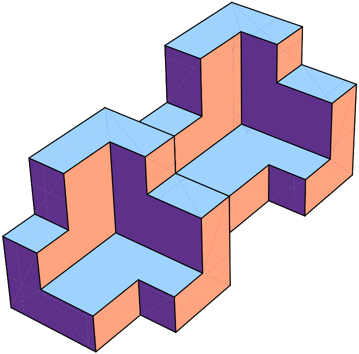

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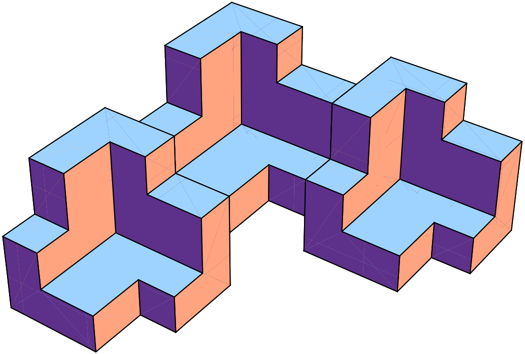

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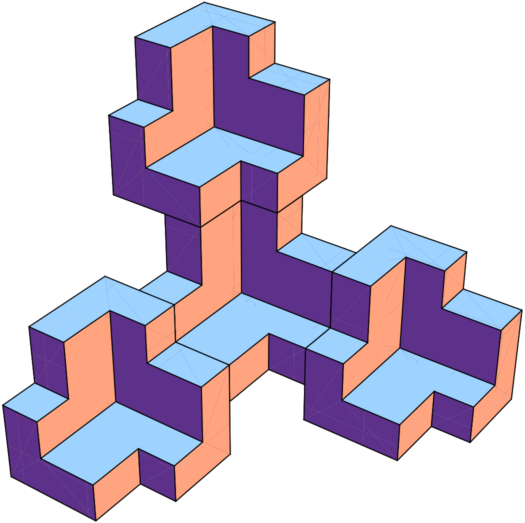

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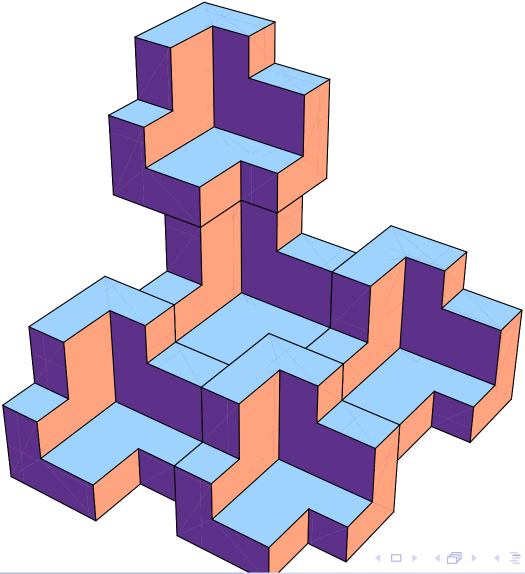

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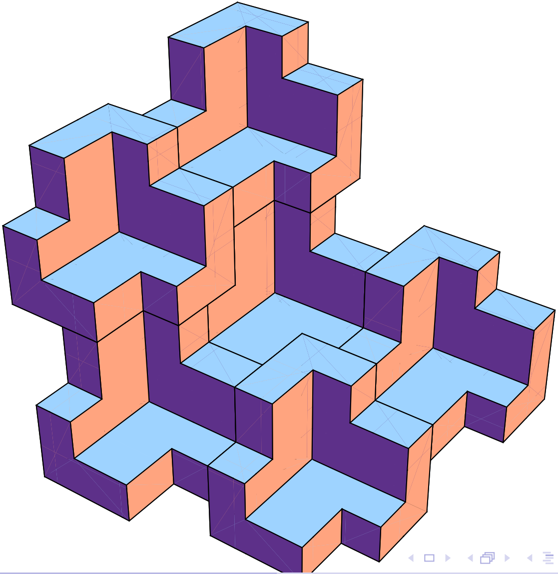

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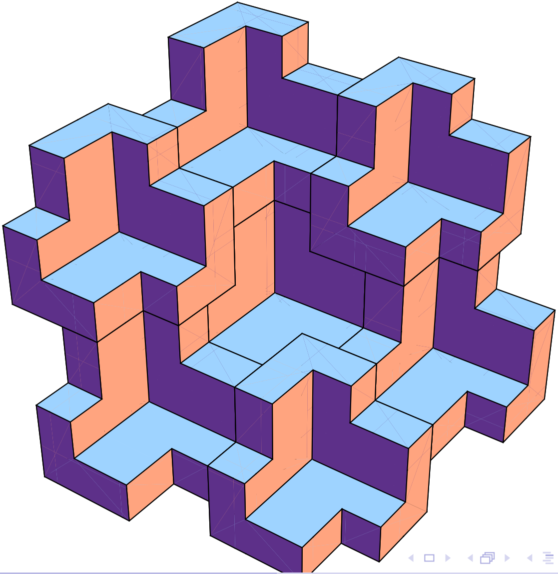

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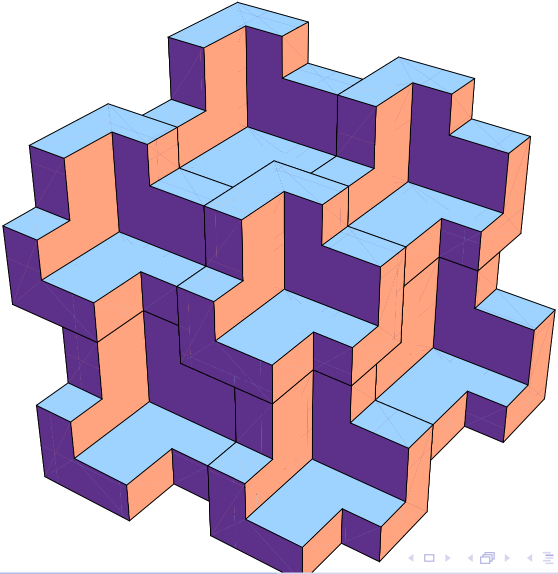

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Less is known on generic results of MDDs when $n \geq 3$. There is no geometrical description of them.

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The set of holes of S is $\overline{S} = \mathbb{N} \setminus S$ and $|\overline{S}| < \infty$ holds.

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The set of **holes** of S is $\overline{S} = \mathbb{N} \setminus S$ and $|\overline{S}| < \infty$ holds.

The **Frobenius number** and the **genus** of S are $\mathfrak{f}(S) = \max \overline{S}$ and $\mathfrak{g}(S) = |\overline{S}|$, respectively.

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Given $t \in S \setminus \{0\}$, the Apéry set of S with respect to t is $\mathrm{Ap}(S, t) = \{s \in S : \ s - t \in \overline{S}\}$.

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$\mathrm{Ap}(S,t)$ has been used for factoring in S.

Introduction

Taking $G = \mathrm{Cay}(\mathbb{Z}_{a_n}, \{a_1, ..., a_{n-1}\}, \{a_1, ..., a_{n-1}\})$ and any related MDD $\mathcal{H}$,

$$\mathrm{Ap}(\langle a_1, ..., a_n\rangle, a_n) = \{\delta(i_1, ..., i_{n-1}) : \ [\![i_1, ..., i_{n-1}]\!] \in \mathcal{H}\}.$$

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Thus, numerical semigroups and Cayley digraphs are connected through their related minimum distance diagrams.

Introduction

Given $m \in S = \langle a_1, ..., a_n \rangle$, a factorization of m in S is $(x_1, ..., x_n) \in \mathbb{N}^n$ such that $x_1 a_1 + \cdots + x_n a_n = m$.

$$\mathcal{F}(m, S) = \{(x_1, ..., x_n) \in \mathbb{N}^n : a_1 x_1 + \cdots + x_n a_n = m\}$$

The denumerant of m in S is $\mathrm{d}(m, S) = |\mathcal{F}(m, S)|$.

Introduction

Example: $m = 87$, $S = \langle 5, 7, 11 \rangle$

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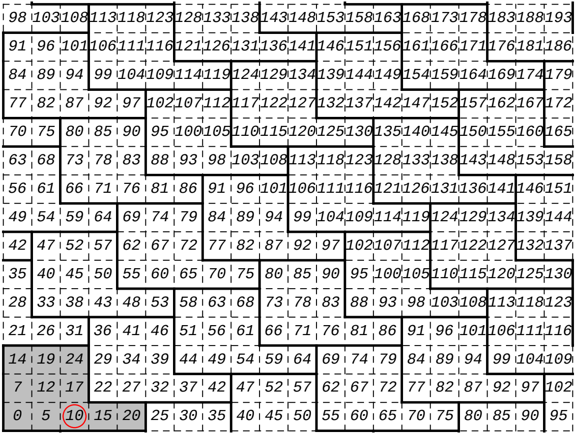

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98	103	108	113	118	123	128	133	138	143	148	153	158	163	168	173	178	183	188	193
91	96	101	106	111	116	121	126	131	136	141	146	151	156	161	166	171	176	181	186
84	89	94	99	104	109	114	119	124	129	134	139	144	149	154	159	164	169	174	179
77	82	87	92	97	102	107	112	117	122	127	132	137	142	147	152	157	162	167	172
70	75	80	85	90	95	100	105	110	115	120	125	130	135	140	145	150	155	160	165
63	68	73	78	83	88	93	98	103	108	113	118	123	128	133	138	143	148	153	158
56	61	66	71	76	81	86	91	96	101	106	111	116	121	126	131	136	141	146	151
49	54	59	64	69	74	79	84	89	94	99	104	109	114	119	124	129	134	139	144
42	47	52	57	62	67	72	77	82	87	92	97	102	107	112	117	122	127	132	137
35	40	45	50	55	60	65	70	75	80	85	90	95	100	105	110	115	120	125	130
28	33	38	43	48	53	58	63	68	73	78	83	88	93	98	103	108	113	118	123
21	26	31	36	41	46	51	56	61	66	71	76	81	86	91	96	101	106	111	116
14	19	24	29	34	39	44	49	54	59	64	69	74	79	84	89	94	99	104	109
7	12	17	22	27	32	37	42	47	52	57	62	67	72	77	82	87	92	97	102
0	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95

$\{(2, 0, 7)\}$

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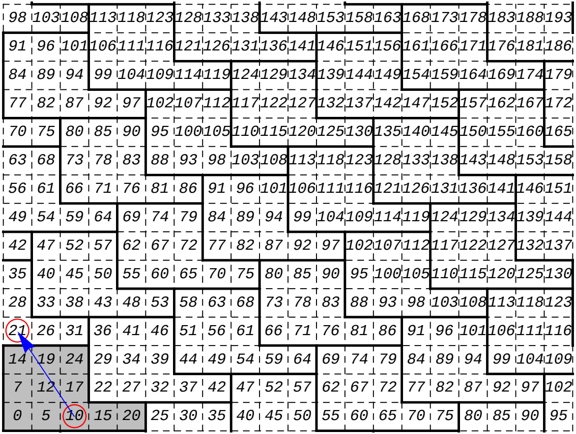

$$\{(2,0,7),(0,3,6)\}$$

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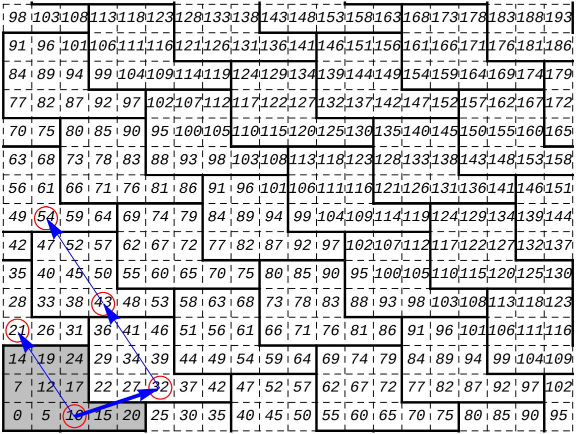

$$\{(2,0,7),(0,3,6),(5,1,5),(3,4,4),(1,7,3)\}$$

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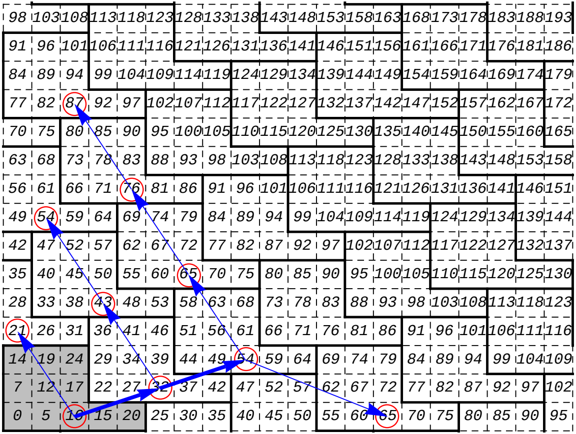

$$\{(2,0,7),(0,3,6),(5,1,5),(3,4,4),(1,7,3),(8,2,3),(13,0,2),$$
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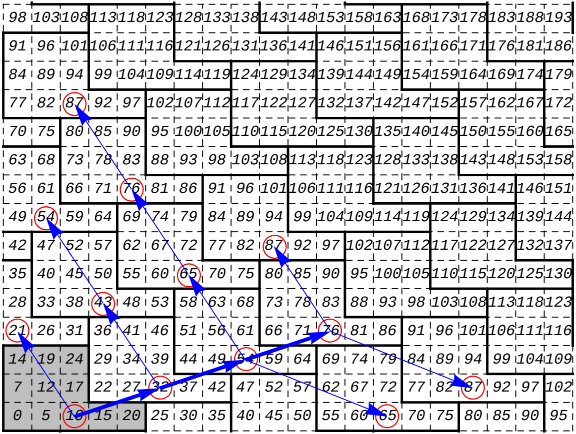

$$\{(2,0,7),(0,3,6),(5,1,5),(3,4,4),(1,7,3),(8,2,3),(13,0,2),\\ (6,5,2),(4,8,1),(2,11,0),(11,3,1),(16,1,0),(9,6,0)\}$$

Minimium path diagram of weighted 2–Cayley digrafs

Minimum path diagram: definition

We are interested in computing the number of minimum paths between any pair of vertices of $G = \mathrm{Cay}(\mathrm{G}_N, \{a, b\}, \{W_a, W_b\})$.

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To this end, we define the minimum path diagram (MPD) associated with G as the unit–square–made region in ${\mathbb{R}_{\geq 0}}^2$, $\mathcal{P}(\mathrm{G}_N, \{a, b\}, \{W_a, W_b\})$, such that it contains each minimum path in G exactly once.

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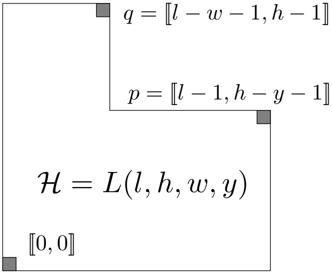
We need a practical description of $\mathcal{P}$ to work with. This description will be given from any MDD $\mathcal{H}$.

Minimum path diagram: definition

Let us assume $\mathcal{H}$ is an MDD related to G.



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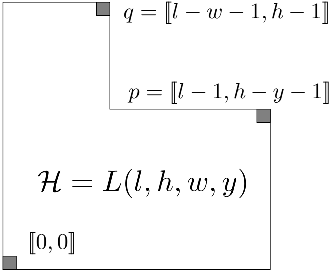

Inequallity $(lW_a - yW_b)(hW_b - wW_a) \geq 0$ holds and both factors don't vanish.

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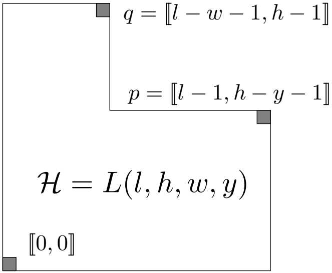

Inequallity $(lW_a - yW_b)(hW_b - wW_a) \geq 0$ holds and both factors don't vanish.

Set $\boldsymbol{u} = (l, -y)$ and $\boldsymbol{v} = (-w, h)$.

Minimum path diagram: characterization

Theorem 1 (characterization of $\mathcal{P}$)

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(a) If either $\delta(p) = \delta(q)$, or $\delta(p) < \delta(q)$ and $lW_a > yW_b$, or $\delta(p) > \delta(q)$ and $hW_b > wW_a$, then

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(c) If $\delta(p)>\delta(q)$ and $hW_b=wW_a$, then

$$\mathcal{P}(\mathrm{G}_N,\{a,b\},\{W_a,W_b\})=\bigcup_{\lambda=0}^{\lfloor\frac{l-1}{w}\rfloor}\Delta(p+\lambda\boldsymbol{v}).$$

Minimum path diagram: characterization

Example 4: $G_1 = \mathrm{Cay}(\mathbb{Z}_3 \times \mathbb{Z}_3, \{(1,0),(0,1)\}, \{1,1\})$

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(0,2)	(1,2)	(2,2)
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(0,0)	(1,0)	(2,0)

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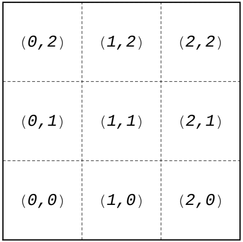

Minimum path diagram: characterization

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(0,2)	(1,2)	(2,2)
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(0,0)	(1,0)	(2,0)

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(0,2)	(1,2)	(2,2)
(0,1)	(1,1)	(2,1)
(0,0)	(1,0)	(2,0)

Minimum path diagram: characterization

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Minimum path diagram: characterization

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10	11										
8	9	10	11								
6	7	8	9	10	11						
4	5	6	7	8	9	10	11				
2	3	4	5	6	7	8	9	10	11		
0	1	2	3	4	5	6	7	8	9	10	11

Minimum path diagram: enumeration

The MPD characterization allow us to enumerate the minimum paths from 0 in G.

Minimum path diagram: enumeration

The MPD characterization allow us to enumerate the minimum paths from 0 in G.

Let us assume that we can efficiently compute the unit square associated with a given vertex u in $\mathcal{H}$, i.e. $u \sim [\![i_0, j_0]\!] \in \mathcal{H}$ (A. and Barguilla, 2008).

Minimum path diagram: enumeration

Theorem 2 (enumeration)

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Given $g \sim [\![i_0, j_0]\!] \in \mathcal{H} = \mathrm{L}(l, h, w, y)$, where $\mathcal{H}$ is an MDD associated with G, let $\mathcal{N}(g, G)$ be the number of minimum paths from 0 to g in G.

Minimum path diagram: enumeration

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(a) if Theorem 1–(a) holds, then $\mathcal{N}(g, G) = \binom{i_0+j_0}{i_0}$,

Minimum path diagram: enumeration

Theorem 2 (enumeration)

Given $g \sim [\![i_0, j_0]\!] \in \mathcal{H} = \mathrm{L}(l,h,w,y)$, where $\mathcal{H}$ is an MDD associated with G, let $\mathcal{N}(g,G)$ be the number of minimum paths from 0 to g in G. Then,

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(b) if Theorem 1–(b) holds, then

$$\mathcal{N}(g,G) = \sum_{\lambda=0}^{\lfloor \frac{j_0}{y} \rfloor} \binom{i_0+j_0+\lambda(l-y)}{j_0-\lambda y},$$

Minimum path diagram: enumeration

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Minimum path diagram: enumeration

Example 6: Minimum paths from $(0,0)$ to $(2,2)$ in $G_3 = \mathrm{Cay}(\mathbb{Z}_3 \times \mathbb{Z}_3, \{(1,0),(0,1)\}, \{1,1\})$

Minimum path diagram: enumeration

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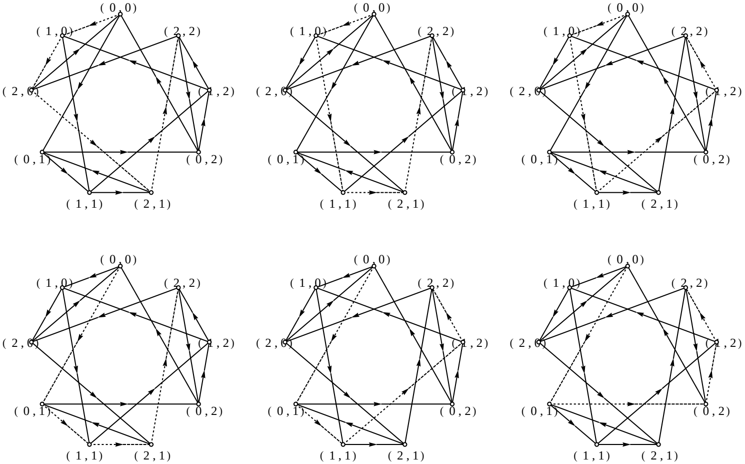

Minimum path diagram: enumeration

Example 6: Theorem 2–(a) holds with $(2,2) \sim [\![2,2]\!]$.

Minimum path diagram: enumeration

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$$\mathcal{N}((2,2),G_3)=\binom{2+2}{2}=6.$$

Minimum path diagram: enumeration

Example 7: $G_4 = \mathrm{Cay}(\mathbb{Z}_{500}, \{1,2\}, \{1,1\})$

Minimum path diagram: enumeration

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#MP to $p = 497$: 249,
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Minimum path diagram: enumeration

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Example 7: $G_5 = \mathrm{Cay}(\mathbb{Z}_{500}, \{1,2\}, \{1,2\})$

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#MP to $p = 497$:
$1394232245616978801397243828704072839500702565876973072641089$
$6294832557162286329069155765887622252129412 5$,
#MP to $q = 499$:
$5325493296145942940693607070474249585412918826163642393957905$
$9478176515507039697978099330699648074089624$.

Maximum genus of numerical 3–semigroups

Genus: general comments

Closed expressions for $\mathfrak{f}(S)$ and $\mathfrak{g}(S)$ are known for 2–semigroups only,

$$\begin{aligned} \mathfrak{f}(\langle a,b\rangle) &= (a-1)(b-1)-1, && \text{Frobenius ?} \\ \mathfrak{g}(\langle a,b\rangle) &= \frac{1}{2}(a-1)(b-1). && \text{Sylvester 1884} \end{aligned}$$

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Curtis 1990 proved the non-existence of polynomial expression of $\mathfrak{f}(S)$ for 3–semigroups.

Genus: general comments

Lewin 1972 gave the following sharp upper bound of $\mathfrak{f}(S)$ for 3–semigroups

$$\mathrm{F}(N) = \max_{\substack{1<a<b<c\leq N \\ \gcd(a,b,c)=1}} \mathfrak{f}(\langle a,b,c\rangle) = \begin{cases} \frac{1}{2}(N-2)^2-1, & N \text{ even}, \\ \frac{1}{2}(N-3)(N-1)-1, & N \text{ odd}. \end{cases}$$

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Erdős and Graham 1972 conjectured that critical semigroups are $S_1=\langle N-2,N-1,N\rangle$ and $S_2=\langle N/2,N-1,N\rangle$ for even $N\geq 4$, and $S_3=\langle (N-1)/2,N-1,N\rangle$ for odd $N\geq 5$.

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Hamidoune 1998 gave more general results for $n \geq 3$.

Genus: problem statement

What can we say about the function

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Not known results.

Genus: problem statement

c	Max $\mathfrak{g}$	Critical semigrups	c	Max $\mathfrak{g}$	Critical semigrups
4	1	$\langle 2,3,4\rangle$	13	30	$\langle 6,12,13\rangle, \langle 11,12,13\rangle$
5	2	$\langle 2,4,5\rangle, \langle 3,4,5\rangle$	14	36	$\langle 7,13,14\rangle, \langle 12,13,14\rangle$
6	4	$\langle 3,5,6\rangle, \langle 4,5,6\rangle$	15	42	$\langle 7,14,15\rangle, \langle 13,14,15\rangle$
7	6	$\langle 3,6,7\rangle, \langle 5,6,7\rangle$	16	49	$\langle 8,15,16\rangle, \langle 14,15,16\rangle$
8	9	$\langle 4,7,8\rangle, \langle 6,7,8\rangle$	17	56	$\langle 8,16,17\rangle, \langle 15,16,17\rangle$
9	12	$\langle 4,8,9\rangle, \langle 7,8,9\rangle$	18	64	$\langle 9,17,18\rangle, \langle 16,17,18\rangle$
10	16	$\langle 5,9,10\rangle, \langle 8,9,10\rangle$	19	72	$\langle 9,18,19\rangle, \langle 17,18,19\rangle$
11	20	$\langle 5,10,11\rangle, \langle 9,10,11\rangle$	20	81	$\langle 10,19,20\rangle, \langle 18,19,20\rangle$
12	25	$\langle 6,11,12\rangle, \langle 10,11,12\rangle$	21	90	$\langle 10,20,21\rangle, \langle 19,20,21\rangle$

Maximum genus and critical semigroups for $c \in \{4,\ldots,21\}$

Genus: solution

Example 8: Given $S = \langle 4, 9, 11 \rangle$, the digraph $\mathrm{Cay}(\mathbb{Z}_{11}, \{4, 9\}, \{4, 9\})$ has related the MDD $\mathrm{L}(5, 3, 4, 1)$.

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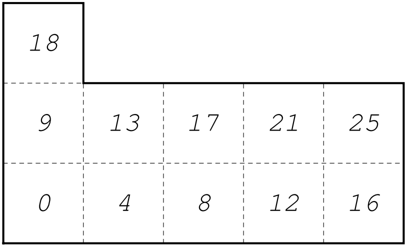

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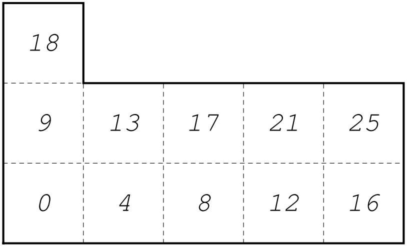

$$\mathrm{Ap}(\langle 4, 9, 11 \rangle, 11) = \{0, 4, 8, 9, 12, 13, 16, 17, 18, 21, 25\}$$

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18				
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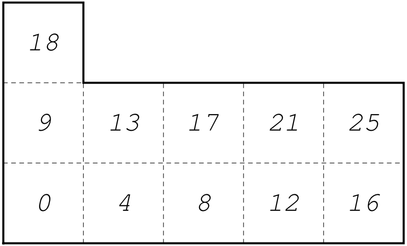

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Selmer expression
$\mathfrak{g}(\langle 4, 9, 11 \rangle) = \frac{0+4+8+9+12+13+16+17+18+21+25}{11} - \frac{11-1}{2} = 8$

Genus: solution

Let $\mathrm{L}(l,h,w,y)$ be an MDD associated with the semigroup $S=\langle a,b,c\rangle$. Then,

$$\begin{aligned}\mathfrak{g}(S) &= \frac{l(h-y)}{2c}[(l-1)a+(h-y-1)b] \\ &\quad + \frac{y(l-w)}{2c}[(l-w-1)a+(2h-y-1)b]-\frac{c-1}{2}. \quad (*)\end{aligned}$$

Let $g(a,b,c,l,h,w,y)$ be defined by $(*)$ on the compact K given by the restrictions $4\leq a+2\leq b+1\leq c\leq N$, $1\leq w+1\leq l\leq c$, $1\leq y+1\leq h\leq c$ and $lh-wy=c$.

Genus: solution

Set $\boldsymbol{x} = (a, b, c, l, h, w, y) \in K$ and consider

$$K = K_1 \cup K_2 \cup U$$

with

$$\begin{aligned} K_1 &= \{\boldsymbol{x} \in K : \; wy = 0\}, \\ K_2 &= \{\boldsymbol{x} \in K : \; wy \geq 1\}, \\ U &= \{\boldsymbol{x} \in K : \; 0 < wy < 1\}. \end{aligned}$$

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Thus, search can be restricted on the compact $K_1 \cup K_2$.

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∂K_1:

(1) For even $N \geq 4$: $g = \frac{1}{4}(N-2)^2$ at
$l = 2, h = N/2, w = 1, y = 0, a = N/2, b = N-1, c = N$ and
$l = N/2, h = 2, w = 1, y = 0, a = N-2, b = N-1, c = N$.

(2) For odd $N \geq 5$: $g = \frac{1}{4}(N-3)(N-1)$ at
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∂K_2:

(3) For even $N \geq 4$: no valid point attains $g = \frac{1}{4}(N-2)^2$.

(4) For odd $N \geq 5$: $g = \frac{1}{4}(N-3)(N-1)$ at
$l = \frac{1}{2}(N+1), h = 2, w = 1, y = 1, a = N-2, b = N-1, c = N$
and $l = 2, h = \frac{1}{2}(N+1), w = 1, y = 1, a = \frac{1}{2}(N-1), b = N-1, c = N$.

Genus: solution

Theorem

$$\mathrm{G}(N) = \begin{cases} \frac{1}{4}(N-2)^2, & N \text{ even}, \\ \frac{1}{4}(N-3)(N-1), & N \text{ odd}. \end{cases}$$

Critical semigroups are almost those of $\mathrm{F}(N)$:

(i) $S_1 = \langle N-2, N-1, N\rangle$ and $S_2 = \langle N/2, N-1, N\rangle$ for even $N \geq 4$.

(ii) S_1 and $S_3 = \langle (N-1)/2, N-1, N\rangle$ for odd $N \geq 5$.