An overview on $\{C_3,\ldots,C_s\}$-free extremal graphs

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Extremal Problems

Given a number *n* and a family of graphs *Ω*, the objective is to obtain:

The maximum number of edges *e* such that any graph on *n* nodes and *e* edges has no subgraph belonging to *Ω*.

$$ex(n;\Omega) = \text{extremal number.}$$

$$EX(n;\Omega) = \{G \text{ with } n \text{ vertices and } ex(n;\Omega) \text{ edges}\}$$

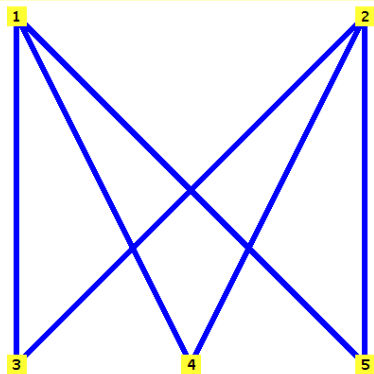
Classic Problem: Ω = {Triangle}

Mantel (1927):

$$ex(n;K_3) = \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil$$

K_3=C_3= 

$$EX(n;K_3) = K_{\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil}$$

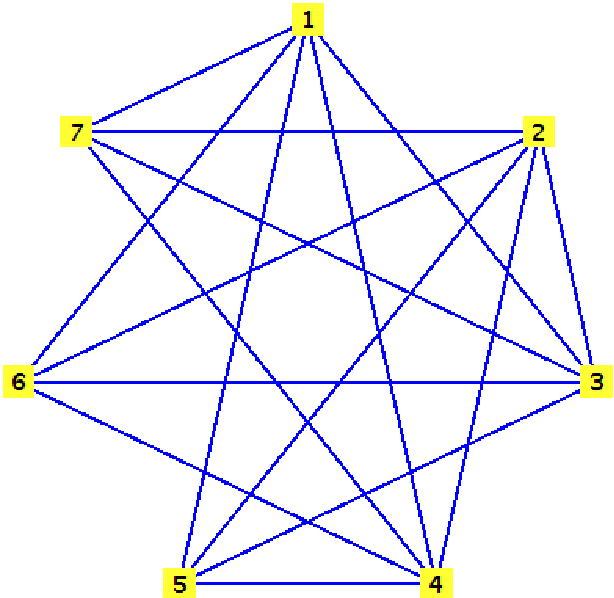


Ω = {Complete on p+1 vertices}

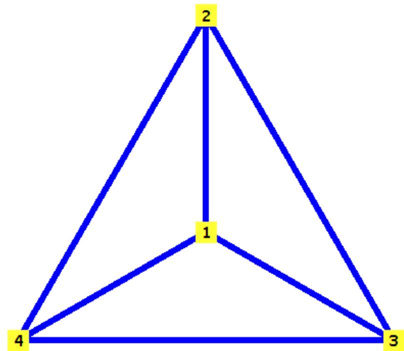
Turàn (1940):

$$ex(n; K_{p+1}) = |E(T_{n,p})|$$

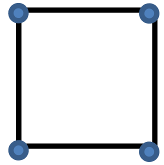
$$EX(n; K_{p+1}) = T_{n,p}$$



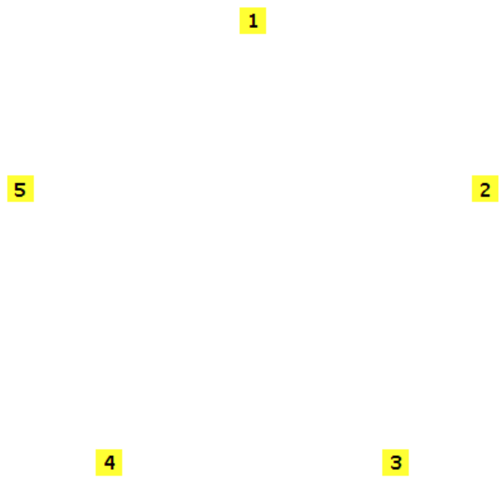
$T_{7,3}$ free of K_4



$\Omega = \{C_4\}$



Example: **n=5** vertices. Find a graph **without squares** having **maximum number of edges**.



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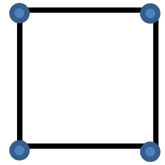

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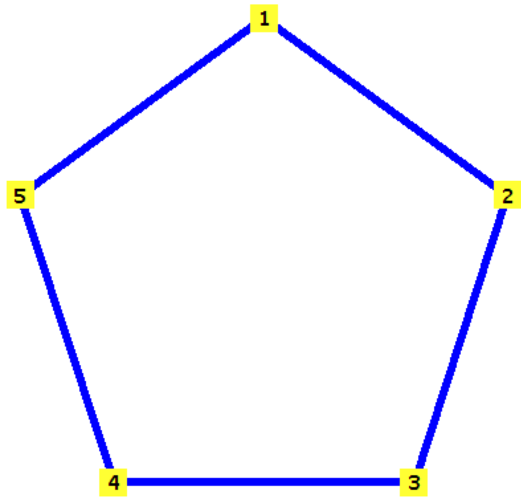

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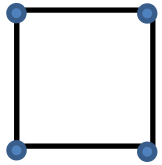

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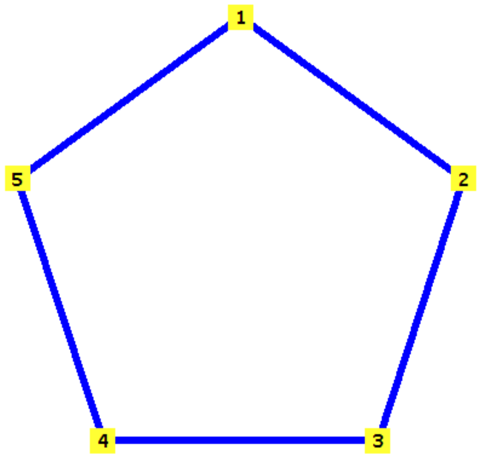

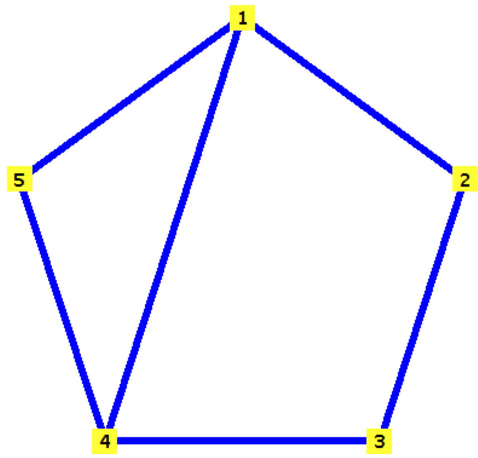

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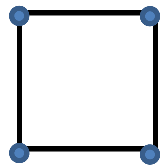

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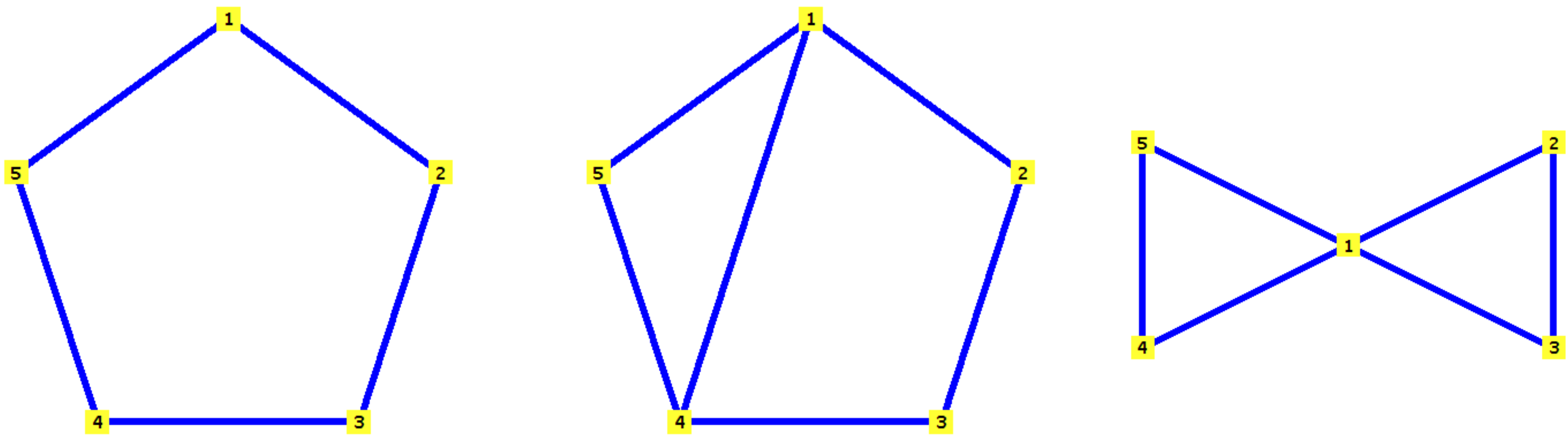


$ex(5;C_4)=6$.

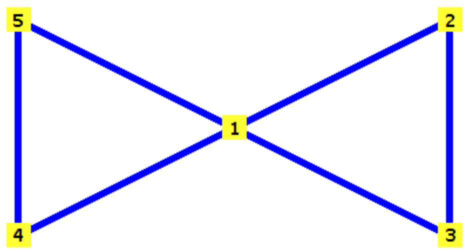
$\Omega = \{C_4\}$

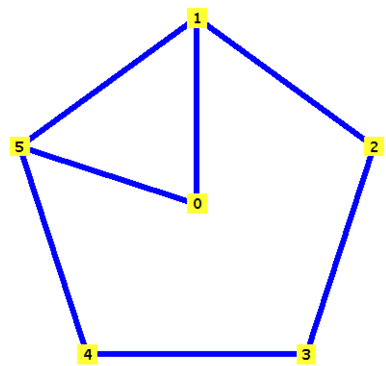
Reymann (1958):

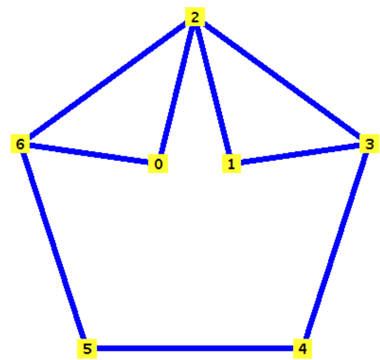
$$ex(n;C_4) \leq \frac{n + n\sqrt{4n-3}}{4}$$

Clapham, Flokhart , Sheehan (1989), Yuansheng , Rowlinson (1992):

Exact value of $ex(n;C_4)$ for $n \leq 31$.







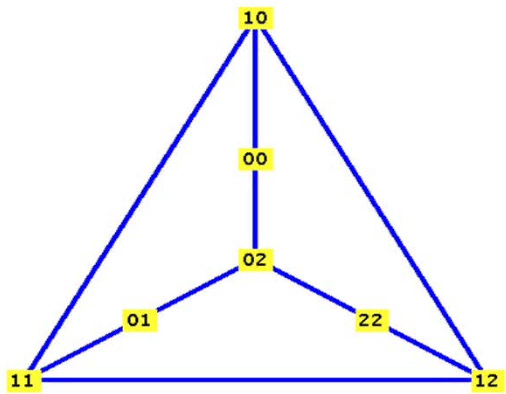
$\Omega = \{C_4\}$

Furedi (1989, 1996): *q* a prime power

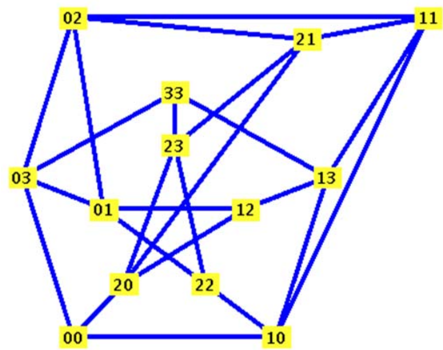
$$ex(q^2 + q + 1; C_4) = \frac{1}{2} q(q+1)^2$$

Erdös, Renyi, Sós (1966), Brown (1966): Using ***polarity graphs*** for proving the lower bound.

q=2



q=3



$\Omega = \{C_4\}$

[ABL2010]: *q* a prime power

There exist C_4-free graphs on $q^2+q+1-h$ vertices, $h = 1, 2, \ldots, 7$ and sizes as follows:

$ex(q^2+q+1-h; C_4) \geq$	$\frac{1}{2}q(q+1)^2 - hq$	$h = 1, 2$	*every* q
	$\frac{1}{2}q(q+1)^2 - 3q + 1$	$h = 3$	*every* q
	$\frac{1}{2}q(q+1)^2 - 4q + 1$	h=4	*odd* q
	$\frac{1}{2}q(q+1)^2 - 4q + 2$		*even* q
	$\frac{1}{2}q(q+1)^2 - hq + h - 3$	h=5,6	*odd* q
	$\frac{1}{2}q(q+1)^2 - hq + h - 2$		*even* q
	$\frac{1}{2}q(q+1)^2 - 7q + 4$	h=7	*odd* q *or* $q = 4$
	$\frac{1}{2}q(q+1)^2 - 7q + 5$		*even* q, $q > 4$

$\Omega = \{C_4\}$

Firke, Kosek, Nash, Williford (2013):

For q an even prime power:

$$ex(q^2 + q; C_4) \le \frac{1}{2} q(q+1)^2 - q \text{ for } q \text{ even.}$$

$$ex(q^2 + q; C_4) = \frac{1}{2} q(q+1)^2 - q$$

Open Problem

To improve upper bounds and **to find lower bounds**.

Tool: Families of graphs free of even cycles have been constructed using methods based on random graphs or on incidence graphs of {finite geometries, some hypergraphs, etc}

$\Omega = \{C_{2t}\}$ *(even cycle)*

Bondy and Simonovits (1974):

$$ex(n; C_{2t}) \leq 100\, t\, n^{1+1/t}$$

Current fashion and open problem

Simonovits, Lazebnik, Ustimenko, Woldar, Furedi, Kostoshka, Alon, etc:

$$\text{To study} \quad ex(n; C_6)$$

From now on …

From now on …

To study the extremal function:

$$ex\left(n;\{C_3, C_4, \ldots, C_s\}\right) = f_s(n),\ s \geq 4.$$

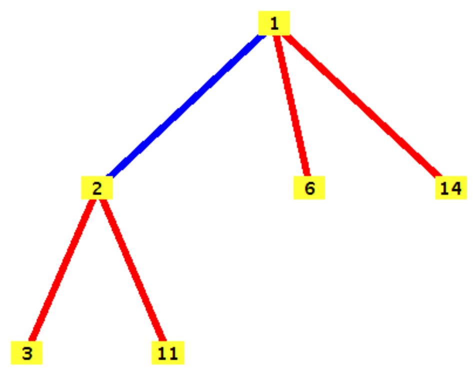
- Structural properties for s≥4.
- Some exact values for small *n* in comparison with s.
- Some results for s=4,5,6,7,11.
- Asymptotic results for s=5,6,7,10,11.

Structural properties

[BCDGv2008]->

EveryG ∈ EX(n;{C_3,C_4,…,C_s}) has **diam(G)≤s-1.**

Therefore λ**(G) =** δ**(G).**



ξ(uv) = d(u) + d(v) -2

Minimum edge degree = ξ(G).

[TLBM2009]-> λ'**(G) = ξ(G).**

Structural properties

Garnick, Nieuwejaar 1992:

What is the girth of a graph *$G \in EX(n;\{C_3,C_4,\ldots,C_s\})$?*

***$g=s+1$* or it could be greater?**

There are some $G \in EX(n;\{C_3,C_4,\ldots,C_s\})$ with girth **$g \geq s+2$**:

$f_6(15)=18$->

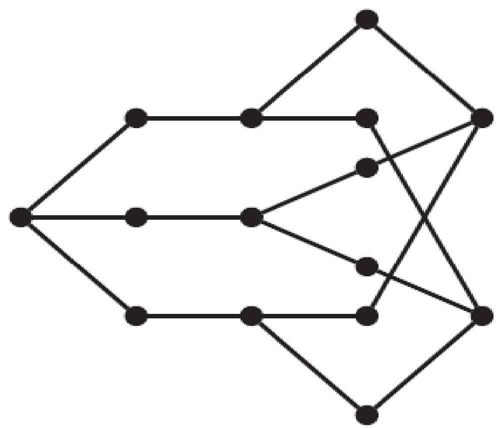


Fig. 1. An extremal graph of $EX(15;\{C_3,\ldots,C_6\})$ having girth 8.

Structural properties: the girth

s=3	$n \geq 4$	[17]
s=4	$n = 5$ and $n \geq 7$	[11]
s=5	$n = 6$ and $n \geq 8$	[15]
s=6	$n \in \{7, 10, 12, 13, 14\}$ and $n \geq 16$	[1, 6, 16]
s=7	$n \geq 14$	[2]

Table 1: For the given values of s, n, all graphs in $EX(n; \{C_3, \ldots, C_s\})$ have girth $s+1$.

[17] Mantel 1927.

[11] Garnick and Nieuwejaar 1992.

[15] Lazebnik and Ping Wang 1997.

[1] Abajo, B, Diánez, 2010.

[6] B, Cera, Diánez, García-Vázquez, 2008.

[16] Tang, Lin, B, Miller, 2009.

[2] Abajo, B, Diánez, 2012.

Structural properties: the girth

The girth ***g*** of G Є EX(n;{C_3,C_4,…,C_s}) is *g=s+1* if:

[LW1997]-> s ≥ 12 and

$$n \geq 2^{a^2+a+1} s^a \text{ for } a = s - 3 - \lfloor (s-2)/4 \rfloor$$

[BGv2007]-> s ≥ 7 and

$$n \geq \frac{2(s-2)^{s-2}+s-5}{s-3} + 1$$

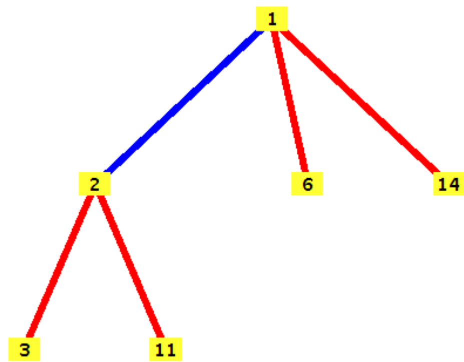
Structural properties: the girth

The girth of G Є $EX(n;\{C_3,C_4,\ldots,C_s\})$ is g=s+1 if

[LW1997]-> $\Delta(G) \geq s$.

[ABD2012]->

- -> $\Xi(G) \geq s-1$.
- -> $\bar{d} > 2s$



Structural properties

[ABD2012]->

Theorem 1. *Let $s \geq 4$ be an integer. Then,*

$$\max\Big\{g(G) : G \in \mathrm{EX}(n; \{C_3, \ldots, C_s\}), n \geq s+1\Big\} = \min\{n, \lfloor 3s/2 \rfloor\}.$$

Exact values: $\Omega = \{C_3,\ldots,C_s\}$

[BDCGv2008]-> For $s \geq 4$, if $s+1 \leq n \leq 3s/2$, then $f_s(n)=n$.

[A2009]-> For $s \geq 4$:

- If $3s/2+1 \leq n \leq 2s-1$, *then* $f_s(n)=n+1$.
- If $2s \leq n \leq 9s/4-1$, *then* $f_s(n)=n+2$.
- If $9s/4 \leq n \leq (8s-2)/3-1$, *then* $f_s(n)=n+3$.
- In particular for $s=4$:

	0	1	2	3	4	5	6	7	8	9
0			1	2	3	5	6	8	10	12
10	15									

Exact values for s=5,7,11.

For s=5:

	0	1	2	3	4	5	6	7	8	9
0			1	2	3	4	6	7	9	10
10	12	14	16	18	**21**					

- For s=7:

	0	1	2	3	4	5	6	7	8	9
0			1	2	3	4	5	6	8	9
1	10	12	13	14	16	18	19	20	22	24
2	25	27	29	30	32	34	36	38	40	42
3	**45**									

Exact values for s=5,7,11

[AD2012]-> *For s=5,7,11,*

(q+1;g)-cages with g=6,8,12 and q a prime power belong to :

$$EX(n;\ \{C_3, C_4, \ldots, C_s\})$$

where $n = 2(1+q+q^2+\ldots+q^{g/2-1})$ *and* $s=g-1$.

The tool was:

Lower bound for irregular graphs

Alon, Hoory and Linial (2002):

Every graph G with girth g and average degree $\bar{d} = \frac{2e}{n}$

$$|V(G)| = n \geq n_0(\bar{d}; g) = \begin{cases} 1 + \bar{d} \sum_{i=0}^{(g-3)/2} (\bar{d} - 1)^i \text{ for g odd} \\ 2 \sum_{i=0}^{g/2-1} (\bar{d} - 1)^i \text{ for g even} \end{cases}$$

If **δ = Δ = k**, then G is **k**-regular and the lower bound is the minimum possible order of a **(k,g)-cage**.

Case s=4: Ω = {C_3,C_4}

Erdös (1975) -> To find $ex(n;\{C_3,C_4\}) = f_4(n)$

Upper bound:

$$f_4(n) \le \frac{n\sqrt{n-1}}{2}$$

⇓

$$\frac{1}{2\sqrt{2}} \le \limsup_{n\to\infty} \frac{f_4(n)}{n^{3/2}} \le \frac{1}{2}$$

$\Omega = \{C_3, C_4\}$

Garnick, Nieuwejaar (1992), Garnick, Kwong, Lazebnik (1993), Wang, Dueck, McMilland (2001)->

Exact values of $f_4(n)$ for **n≤30.**

They also prove that $f_4(50) = 175$

$$EX(50; \{C_3, C_4\}) = Hoffmann-Singleton\,Graph$$

They provide a **table with lower bounds for *n≤200*.**

- Method: algorithms combining hill-climbing, backtracking o simulated annealing techniques.

✓ **Abajo, B, Diánez (2010): We i**mprove a lot of lower bounds by means of a theoretical result for an infinite family of values.

n	0	1	2	3	4	5	6	7	8	9
0	**0**	**0**	**1**	**2**	**3**	**5**	**6**	**8**	**10**	**12**
10	**15**	**16**	**18**	**21**	**23**	**26**	**28**	**31**	**34**	**38**
20	**41**	**44**	**47**	**50**	**54**	**57**	**61**	**65**	**68**	**72**
30	**76**	80	85	87	90	94	99	104	109	114
40	120	124	129	134	139	144	150	156	162	168
50	**175**	176	178	181	185	188	192	195	199	203
60	207	212	216	221	226	231	235	240	245	250
70	255	260	265	270	275	280	285	291	296	301
80	306	311	317	323	329	334	340	346	352	357
90	363	368	374	379	385	391	398	404	410	416
100	422	428	434	440	448	456	458	464	470	476
110	483	489	495	501	508	514	520	526	532	538
120	544	551	558	565	571	578	584	592	600	608
130	616	624	632	640	648	657	666	669	672	675
140	679	686	693	700	707	714	721	728	735	742
150	749	756	763	770	777	784	792	801	810	819
160	828	838	847	856	865	874	883	892	901	910
170	920	930	932	935	938	941	944	948	956	963
180	971	979	986	994	1001	1009	1017	1025	1033	1041
190	1049	1057	1065	1073	1081	1089	1097	1105	1113	1121
200	1129									

Table 1: Known lower bounds on $ex(n; \{C_3, C_4\})$, from tables printed in [1, 7, 8, 14]. Exact values, when known, are listed in bold font.

$$\Omega = \{C_3, C_4\}$$

Theorem [BMM14]->

$$f_4(31) = 80$$
$$f_4(32) = 85$$

[BMM14]-> Algorithm "Grow and Prune (GAP)"

- **Start with *G* of *g=5* and as many edges as possible**.
- **Delete a vertex *x* of minimum degree delta**:
  - If there is ***x*** such that **diam(G-x)=4**, delete **x** and add as many edges as necessary to obtain a graph on diameter**3**.
  - If **diam(G-x)=3** for all ***x***, delete x_0 such that the minimum degree of ***G-*** x_0 is **delta-1**.
  - If the minimum degree of ***G-x*** is **delta** for all ***x***, choose arbitrarily ***x*** of minimum degree **delta**.

n	0	1	2	3	4	5	6	7	8	9
30		80	85			95				
40						145				
50										
60										251
70	257	263	269	275	281	288	294	301	307	314
80	320	326	332	339	346	352	359	366	373	381
90	388	395	402	409	417	424	432	440	449	452
100	455	459	462	466	469	475	482	489	497	505
110	512	520	528	536	544	552	560	567	575	583
120	591	598	607	615	623	631	640	649	658	661
130	665	668	672	675	683	691	700	708	717	725
140	734	743	752	761	770	779	788	797	806	815
150	825	834	843	853	863	872	882	892	902	912
160	923	934	945	948	951	954	957	961	964	968
170	972	975	979	983	987	992	997	1002	1006	1011
180	1016	1023	1033	1043	1052	1061	1071	1080	1089	1099
190	1108	1118	1128	1138	1148	1158	1168	1178	1188	1198
200	1208	1218	1228	1238	1248	1258	1268	1278	1288	1298
210	1308	1318	1328	1338	1348	1359	1370	1381	1392	1402
220	1412	1423	1434	1445	1456	1467	1478	1489	1500	1511
230	1522									

Table 4: Improved lower bounds on $ex(n; \{C_3, C_4\})$ produced by application of our GaP algorithm.

$\Omega = \{C_3, C_4\}$

Theorem [ABD2010]-> q a prime power

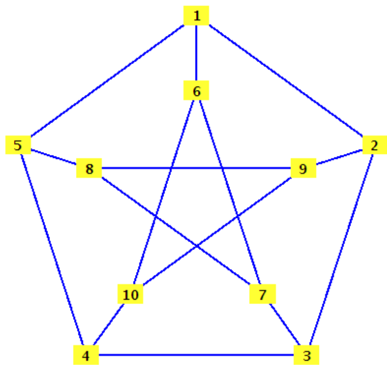
$$f_4(2q^2+q) \geq q^2(q+1)+(q+1)f_4(q).$$

$For\ q \leq 11:$

$$f_4(2q^2+q-h) \geq q^2(q+1)+(q+1)f_4(q)-hq+\varepsilon, \varepsilon = \begin{cases} -h, & h=1,2 \\ -2, & h \geq 3 \end{cases}$$

$$for\ q \geq 13: \ f_4(2q^2+q-h) \geq q^2(q+1)+(q+1)f_4(q)-h(q+1)$$

$f_4(10)=15$,



$f_4(21)=44$, $f_4(20)=41$.

$\Omega = \{C_3, C_4\}$

Theorem [AB2014, in preparation]-> *q* a prime power

$$f_4(2q^2+q-1) \geq q^2(q+1)+(q+1)f_4(q)+f_4(q-1)-q;$$
$$f_4(2q^2+q-h) \geq q^2(q+1)+(q+1)f_4(q)+f_4(q-h)-q, \text{ for } h=2,\dots,q.$$

For h=1, the lower bound contained in Theorem [ABD10] is improved in $f_4(q-1)+1$.

For h≥2, the lower bound contained in Theorem [ABD10] is improved in $f_4(q-h)+h(q+1)-q$.

Case s=5: HSAGA [TLBM09]

Table 1

The current lower bounds for $ex(v; \{C_3, C_4, C_5\})$ for $v \le 39$

N. of vertices	0	10	20	30
0	\	$e = 12$ $\mathcal{D} = \{2_6, 3_4\}$	$e = 34$ $\mathcal{D} = \{3_{12}, 4_8\}$	$e = 61$ $\mathcal{D} = \{3_2, 4_{24}, 5_4\}$
1	\	$e = 14$ $\mathcal{D} = \{2_5, 3_6\}$	$e = 36$ $\mathcal{D} = \{2_1, 3_{10}, 4_{10}\}$	$e = 63$ $\mathcal{D} = \{3_3, 4_{23}, 5_5\}$ 64
2	\	$e = 16$ $\mathcal{D} = \{2_4, 3_8\}$	$e = 39$ $\mathcal{D} = \{3_{10}, 4_{12}\}$	$e = 67$ $\mathcal{D} = \{4_{26}, 5_6\}$
3	\	$e = 18$ $\mathcal{D} = \{2_3, 3_{10}\}$	$e = 42$ $\mathcal{D} = \{3_8, 4_{15}\}$	$e = 69$ $\mathcal{D} = \{3_3, 4_{21}, 5_9\}$ 70
4	\	$e = 21$ $\mathcal{D} = \{3_{14}\}$	$e = 45$ $\mathcal{D} = \{3_6, 4_{18}\}$	$e = 73$ $\mathcal{D} = \{3_3, 4_{18}, 5_{13}\}$ 74
5	\	$e = 22$ $\mathcal{D} = \{2_1, 3_{14}\}$	$e = 48$ $\mathcal{D} = \{3_4, 4_{21}\}$	$e = 74$ $\mathcal{D} = \{3_3, 4_{21}, 5_{11}\}$ 77
6	$e = 6$ $\mathcal{D} = \{2_6\}$	$e = 24$ $\mathcal{D} = \{3_{16}\}$	$e = 52$ $\mathcal{D} = \{4_{26}\}$	$e = 76$ 81 $\mathcal{D} = \{2_1, 3_2, 4_{23}, 5_8, 6_2\}$
7	$e = 7$ $\mathcal{D} = \{1_1, 2_5, 3_1\}$	$e = 26$ $\mathcal{D} = \{2_2, 3_{12}, 4_3\}$	$e = 53$ $\mathcal{D} = \{2_1, 3_1, 4_{24}, 5_1\}$	$e = 82$ $\mathcal{D} = \{3_1, 4_{19}, 5_{17}\}$ 84
8	$e = 9$ $\mathcal{D} = \{2_6, 3_2\}$	$e = 29$ $\mathcal{D} = \{3_{14}, 4_4\}$	$e = 55$ $\mathcal{D} = \{2_1, 3_2, 4_{23}, 5_2\}$	$e = 85$ 88 $\mathcal{D} = \{2_1, 3_1, 4_{16}, 5_{19}, 6_1\}$
9	$e = 10$ $\mathcal{D} = \{2_7, 3_2\}$	$e = 31$ $\mathcal{D} = \{2_1, 3_{12}, 4_6\}$	$e = 57$ $\mathcal{D} = \{3_5, 4_{21}, 5_3\}$	$e = 89$ 92 $\mathcal{D} = \{3_1, 4_{15}, 5_{23}\}$

Case s=7: HSAGA [TLBM09]

Table 3
The current lower bounds for $ex(v; \{C_3, C_4, C_5, C_6, C_7\})$ for $v \leq 39$

N. of vertices	0	10	20	30
0	\	$e = 10$ $\mathcal{D} = \{1_2, 2_6, 3_2\}$	$e = 25$ $\mathcal{D} = \{2_{10}, 3_{10}\}$	$e = 45$ $\mathcal{D} = \{3_{30}\}$
1	\	$e = 12$ $\mathcal{D} = \{2_9, 3_2\}$	$e = 27$ $\mathcal{D} = \{2_9, 3_{12}\}$	$e = 46$ $\mathcal{D} = \{2_1, 3_{30}\}$
2	\	$e = 13$ $\mathcal{D} = \{2_{10}, 3_2\}$	$e = 29$ $\mathcal{D} = \{2_8, 3_{14}\}$	$e = 47$ $\mathcal{D} = \{2_2, 3_{30}\}$
3	\	$e = 14$ $\mathcal{D} = \{1_1, 2_9, 3_3\}$	$e = 30$ $\mathcal{D} = \{1_1, 2_7, 3_{15}\}$	$e = 49$ $\mathcal{D} = \{2_3, 3_{28}, 4_2\}$
4	\	$e = 16$ $\mathcal{D} = \{2_{10}, 3_4\}$	$e = 32$ $\mathcal{D} = \{2_8, 3_{16}\}$	$e = 51$ $\mathcal{D} = \{2_3, 3_{28}, 4_3\}$
5	\	$e = 18$ $\mathcal{D} = \{2_9, 3_6\}$	$e = 34$ $\mathcal{D} = \{2_7, 3_{18}\}$	$e = 52$ $\mathcal{D} = \{2_4, 3_{28}, 4_3\}$ **53**
6	\	$e = 19$ $\mathcal{D} = \{1_1, 2_8, 3_7\}$	$e = 36$ $\mathcal{D} = \{2_6, 3_{20}\}$	$e = 54$ **55** $\mathcal{D} = \{2_3, 3_{23}, 4_9, 5_1\}$
7	\	$e = 20$ $\mathcal{D} = \{1_1, 2_9, 3_7\}$	$e = 38$ $\mathcal{D} = \{2_5, 3_{22}\}$	$e = 56$ $\mathcal{D} = \{2_1, 3_{26}, 4_{10}\}$
8	$e = 8$ $\mathcal{D} = \{2_8\}$	$e = 22$ $\mathcal{D} = \{2_{10}, 3_8\}$	$e = 40$ $\mathcal{D} = \{2_4, 3_{24}\}$	$e = 58$ $\mathcal{D} = \{2_1, 3_{26}, 4_{11}\}$
9	$e = 9$ $\mathcal{D} = \{1_1, 2_7, 3_1\}$	$e = 24$ $\mathcal{D} = \{2_9, 3_{10}\}$	$e = 42$ $\mathcal{D} = \{2_3, 3_{26}\}$	$e = 59$ **60** $\mathcal{D} = \{2_2, 3_{26}, 4_{11}\}$

Upper bounds for s=5,6

[ABD2010]->

Corollary 6. *For $n \geq 7$ the following upper bounds hold:*

(i) $f_5(n) \leq \frac{n}{4}(1+\sqrt{2n-3})$;

(ii) $f_6(n) \leq \frac{1}{12}\left(c-\frac{8}{c}+2\right)n$, *where*

$$c = \left(-136+108n+12\sqrt{132-204n+81n^2}\right)^{1/3}.$$

Asymptotics for s=5,6,7,10,11

Theorem [ABD2012]->

$$\limsup_{n\to\infty} \frac{f_s(n)}{n^{1+\frac{2}{s-1}}} = \frac{1}{2^{1+\frac{2}{s-1}}} \text{ for } s = 5,7,11;$$

$$\frac{1}{2^{1+\frac{2}{s}}} \le \limsup_{n\to\infty} \frac{f_s(n)}{n^{1+\frac{2}{s}}} \le \frac{1}{2} \text{ for s} = 6,10.$$

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