Transversal extensions of transversal matroids

Anna de Mier

Universitat Politècnica de Catalunya

Joint work with:

Joseph Bonin, George Washington University

What is it about?

- Take a transversal matroid M on the ground set E
- Let N be a transversal matroid on $E \cup x$ such that $N \backslash x = M$
- We would like to know about N

What is it about?

- Take a transversal matroid M on the ground set E
- Let N be a transversal matroid on $E \cup x$ such that $N \backslash x = M$
- We would like to know about N

  For instance:
  - How do we obtain one/all such N?
  - Are there many such N's?
  - How do different N's relate to each other?

Warm up

Let E be a finite set. Do the following:

- Take a collection $A_1, \ldots, A_r$ with $A_i \subseteq E$, $1 \leq i \leq r$

Warm up

Let E be a finite set. Do the following:

- Take a collection $A_1, \ldots, A_r$ with $A_i \subseteq E$, $1 \leq i \leq r$
- Take an r-simplex Δ with vertices $v_1, \ldots, v_r$

Warm up

Let E be a finite set. Do the following:

- Take a collection $A_1, \ldots, A_r$ with $A_i \subseteq E$, $1 \leq i \leq r$
- Take an r-simplex Δ with vertices $v_1, \ldots, v_r$
- For each $e \in E$, place e on the face spanned by $\{v_i : e \in A_i\}$ as freely as possible

Warm up

Let E be a finite set. Do the following:

- Take a collection $A_1, \ldots, A_r$ with $A_i \subseteq E$, $1 \leq i \leq r$
- Take an r-simplex Δ with vertices $v_1, \ldots, v_r$
- For each $e \in E$, place e on the face spanned by $\{v_i : e \in A_i\}$ as freely as possible

Claim: $\{e_1, \ldots, e_k\}$ is affinely independent if there are $i_1, \ldots, i_k$ all different such that $e_j \in A_{i_j}$ for $1 \leq j \leq k$

Warm up

Let E be a finite set. Do the following:

- Take a collection $A_1, \dots, A_r$ with $A_i \subseteq E$, $1 \leq i \leq r$
- Take an r-simplex Δ with vertices $v_1, \dots, v_r$
- For each $e \in E$, place e on the face spanned by $\{v_i : e \in A_i\}$ as freely as possible

Claim: $\{e_1, \dots, e_k\}$ is affinely independent if there are $i_1, \dots, i_k$ all different such that $e_j \in A_{i_j}$ for $1 \leq j \leq k$

(it follows from Hall's theorem)

Transversal matroids

Let $A_1, \ldots, A_r$ be subsets of a finite set E

A subset $\{e_1, \ldots, e_k\} \subseteq E$ is a partial transversal of $A_1, \ldots, A_r$ if there are $i_1, \ldots, i_k$ all different such that $e_j \in A_{i_j}$ for all $1 \leq j \leq k$

Thm (Edmonds and Fulkerson 1965)
The partial transversals of $A_1, \ldots, A_r$ are the independent sets of a matroid on E

Matroids

Def A matroid consists of

- a finite non-empty set E (the ground set)
- a family $\mathcal{I}$ of subsets of E (the independent sets)

such that

I.1 $\emptyset \in \mathcal{I}$

I.2 if $I' \subseteq I \in \mathcal{I}$ then $I' \in \mathcal{I}$

I.3 if $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$, then there exists $e \in I_2 \backslash I_1$ such that $I_1 \cup e \in \mathcal{I}$

Matroids

Def A matroid consists of

- a finite non-empty set E (the ground set)
- a family $\mathcal{I}$ of subsets of E (the independent sets)

such that

I.1 $\emptyset \in \mathcal{I}$

I.2 if $I' \subseteq I \in \mathcal{I}$ then $I' \in \mathcal{I}$

I.3 if $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$, then there exists $e \in I_2 \backslash I_1$ such that $I_1 \cup e \in \mathcal{I}$

Example:
E a finite set of points in affine space
$\mathcal{I} = \{I \subseteq E : I \text{ is affinely independent } \}$

Some matroid facts

- Given $X \subseteq E$, all maximal independent sets contained in X have the same size, the *rank* $r(X)$ of X

Some matroid facts

- Given $X \subseteq E$, all maximal independent sets contained in X have the same size, the *rank* $r(X)$ of X
- A hyperplane is a maximal set of rank $r(M) - 1$

Some matroid facts

- Given $X \subseteq E$, all maximal independent sets contained in X have the same size, the *rank* $r(X)$ of X
- A hyperplane is a maximal set of rank $r(M) - 1$
- The complement of a hyperplane is called a cocircuit

Some matroid facts

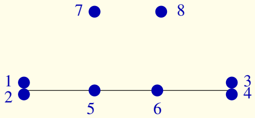
- Given $X \subseteq E$, all maximal independent sets contained in X have the same size, the *rank* $r(X)$ of X
- A hyperplane is a maximal set of rank $r(M) - 1$
- The complement of a hyperplane is called a cocircuit
- If $x \in X$ is such that $r(X - x) = r(X) - 1$, we say that x is a coloop of X (so x is in all maximal independent sets of X)

Some matroid facts

- Given $X \subseteq E$, all maximal independent sets contained in X have the same size, the *rank* $r(X)$ of X
- A hyperplane is a maximal set of rank $r(M) - 1$
- The complement of a hyperplane is called a cocircuit
- If $x \in X$ is such that $r(X - x) = r(X) - 1$, we say that x is a coloop of X (so x is in all maximal independent sets of X)
- Given M and an element $e \in E$, the deletion $M \backslash e$ is the matroid on $E - e$ with independent sets $\{I \in \mathcal{I} : e \notin I\}$

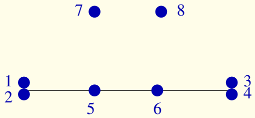
A transversal matroid

$\mathcal{A} = (\{1,2,5,6,7\},\{3,4,5,6,8\},\{7,8\})$ gives



A transversal matroid

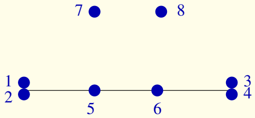
$\mathcal{A} = (\{1, 2, 5, 6, 7\}, \{3, 4, 5, 6, 8\}, \{7, 8\})$ gives



Observe that $(\{1, 2, 5, 6, 7\}, \{3, 4, 5, 6, 7\}, \{7, 8\}$ also gives the same matroid

A transversal matroid

$\mathcal{A} = (\{1,2,5,6,7\},\{3,4,5,6,8\},\{7,8\})$ gives



Observe that $(\{1,2,5,6,7\},\{3,4,5,6,7\},\{7,8\}$ also gives the same matroid

The collections $(\{1,2,5,6,7\},\{3,4,5,6,8\},\{7,8\})$ and $(\{1,2,5,6,7\},\{3,4,5,6,7\},\{7,8\})$ are presentations of the transversal matroid

Representing transversal matroids on a simplex

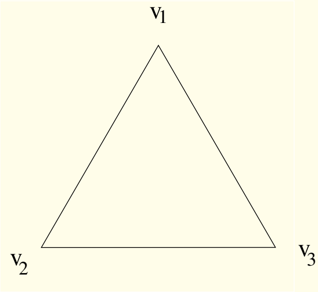
Thm (Brylawski 1975) "Transversal matroids are those that admit a representation in a simplex Δ where elements lie on the faces of Δ is the most free possible way"

Given such a representation, we can recover the presentation as

$$A_i = \{x : x \text{ is not on the face opposite to vertex } v_i\}$$

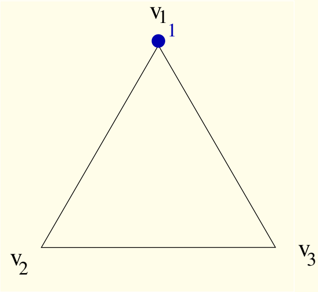
Example again

$$\mathcal{A} = (\{1,2,5,6,7\},\{3,4,5,6,8\},\{7,8\})$$



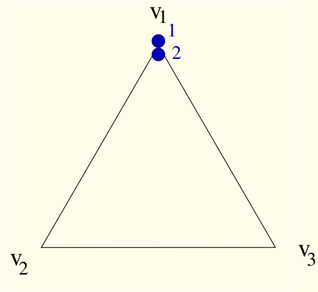
Example again

$$\mathcal{A} = (\{1, 2, 5, 6, 7\}, \{3, 4, 5, 6, 8\}, \{7, 8\})$$



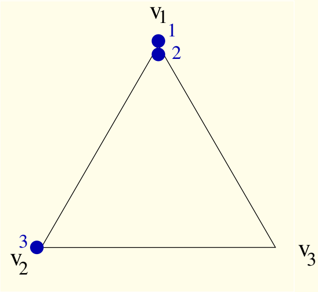
Example again

$$\mathcal{A} = (\{1,2,5,6,7\},\{3,4,5,6,8\},\{7,8\})$$



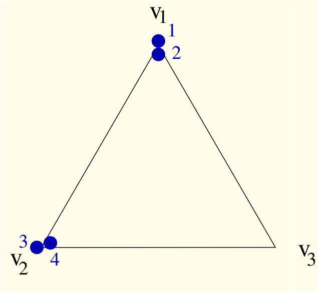
Example again

$$\mathcal{A} = (\{1,2,5,6,7\},\{3,4,5,6,8\},\{7,8\})$$



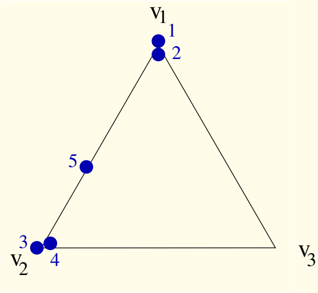
Example again

$$\mathcal{A} = (\{1,2,5,6,7\},\{3,4,5,6,8\},\{7,8\})$$



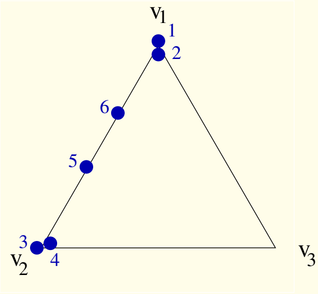
Example again

$$\mathcal{A} = (\{1,2,5,6,7\},\{3,4,5,6,8\},\{7,8\})$$



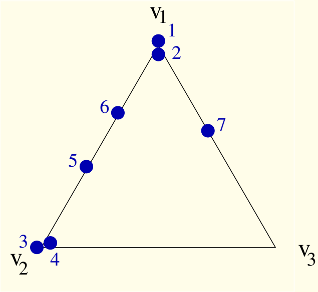
Example again

$$\mathcal{A} = (\{1,2,5,6,7\}, \{3,4,5,6,8\}, \{7,8\})$$



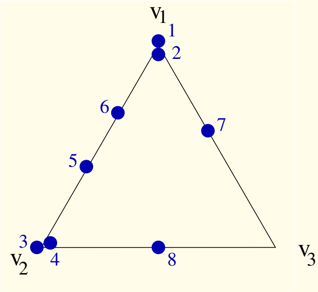
Example again

$$\mathcal{A} = (\{1,2,5,6,7\}, \{3,4,5,6,8\}, \{7,8\})$$



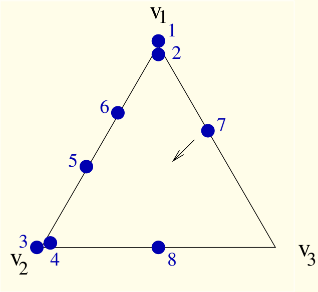
Example again

$$\mathcal{A} = (\{1, 2, 5, 6, 7\}, \{3, 4, 5, 6, 8\}, \{7, 8\})$$



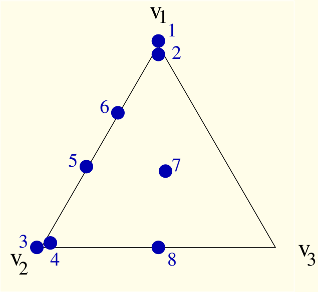
Example again

$$\mathcal{A} = (\{1,2,5,6,7\},\{3,4,5,6,8\},\{7,8\})$$



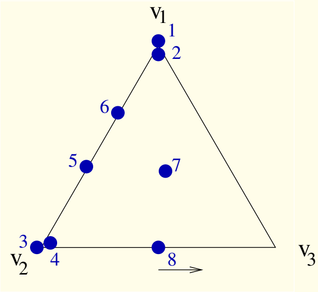
Example again

$$\mathcal{A} = (\{1, 2, 5, 6, 7\}, \{3, 4, 5, 6, 7, 8\}, \{7, 8\})$$



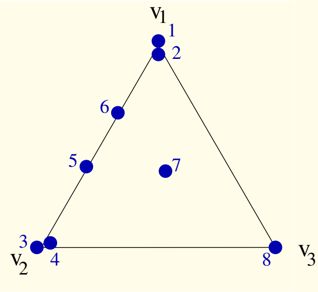
Example again

$$\mathcal{A} = (\{1, 2, 5, 6, 7\}, \{3, 4, 5, 6, 7, 8\}, \{7, 8\})$$



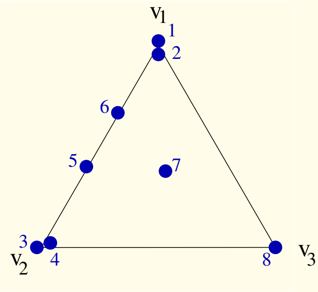
Example again

$$\mathcal{A} = (\{1,2,5,6,7\},\{3,4,5,6,7\},\{7,8\})$$

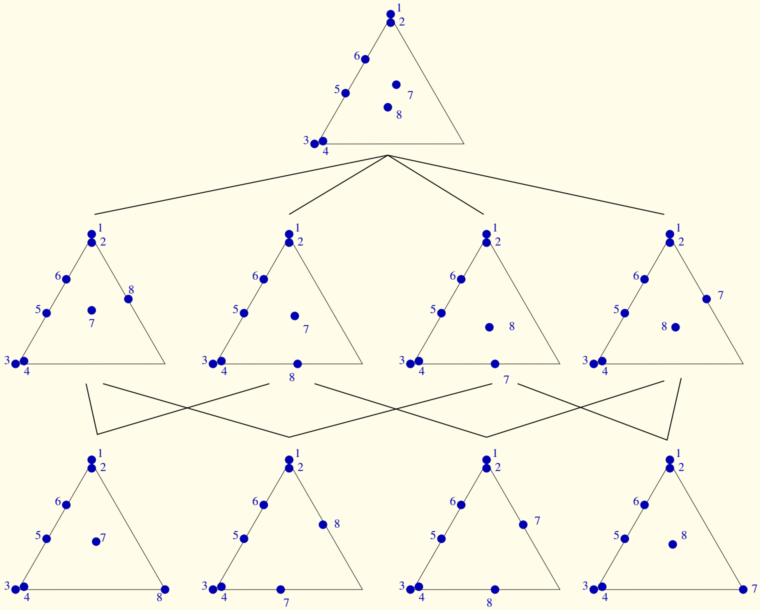


Example again

$$\mathcal{A} = (\{1, 2, 5, 6, 7\}, \{3, 4, 5, 6, 7\}, \{7, 8\})$$



The poset of all presentations



Some facts about presentations

The set of presentations of a transversal matroid is ordered by set inclusion

- A transversal matroid can typically have many minimal presentations
- The sets in a minimal presentation are cocircuits (Las Vergnas 70, Bondy and Welsh 71)

Some facts about presentations

The set of presentations of a transversal matroid is ordered by set inclusion

- A transversal matroid can typically have many minimal presentations
- The sets in a minimal presentation are cocircuits (Las Vergnas 70, Bondy and Welsh 71)
- A transversal matroid has a unique maximal presentation (Mason 69, Bondy 72)
- If $(A_1, A_2, \ldots, A_r)$ and $(B_1, A_2, \ldots, A_r)$ are presentations with $A_1 \subset B_1$, the elements of $B_1 \backslash A_1$ are coloops of $M \backslash A_1$ (Bondy and Welsh 71)

Transversal extensions

A (single-element) extension of a matroid M on E is a matroid N on $E \cup x$ such that $M = N \backslash x$ (and, for us, $r(N) = r(M)$)

- The theory of extensions is well-understood: extensions of M are in bijection with some families of subsets called "modular cuts of flats" (Crapo 65)

Transversal extensions

A (single-element) extension of a matroid M on E is a matroid N on $E \cup x$ such that $M = N \backslash x$ (and, for us, $r(N) = r(M)$)

- The theory of extensions is well-understood: extensions of M are in bijection with some families of subsets called "modular cuts of flats" (Crapo 65)

A transversal extension of a transversal matroid M is an extension of M that is also transversal

Transversal extensions

A (single-element) extension of a matroid M on E is a matroid N on $E \cup x$ such that $M = N \backslash x$ (and, for us, $r(N) = r(M)$)

- The theory of extensions is well-understood: extensions of M are in bijection with some families of subsets called "modular cuts of flats" (Crapo 65)

A transversal extension of a transversal matroid M is an extension of M that is also transversal

- If $(B_1, \ldots, B_r)$ is a presentation of N, then $(B_1 \backslash x, \ldots, B_r \backslash x)$ is a presentation of $M = N \backslash x$

Transversal extensions

A (single-element) extension of a matroid M on E is a matroid N on $E \cup x$ such that $M = N \backslash x$ (and, for us, $r(N) = r(M)$)

- The theory of extensions is well-understood: extensions of M are in bijection with some families of subsets called "modular cuts of flats" (Crapo 65)

A transversal extension of a transversal matroid M is an extension of M that is also transversal

- If $(B_1, \ldots, B_r)$ is a presentation of N, then $(B_1 \backslash x, \ldots, B_r \backslash x)$ is a presentation of $M = N \backslash x$
- So all transversal extensions of M can be obtained by adding x to some sets in some presentation of M.

Transversal extensions

A (single-element) extension of a matroid M on E is a matroid N on $E \cup x$ such that $M = N \backslash x$ (and, for us, $r(N) = r(M)$)

- The theory of extensions is well-understood: extensions of M are in bijection with some families of subsets called "modular cuts of flats" (Crapo 65)

A transversal extension of a transversal matroid M is an extension of M that is also transversal

- If $(B_1, \ldots, B_r)$ is a presentation of N, then $(B_1 \backslash x, \ldots, B_r \backslash x)$ is a presentation of $M = N \backslash x$
- So all transversal extensions of M can be obtained by adding x to some sets in some presentation of M.
- But could it be that we get repetitions? How can we ensure we have all extensions?

Notation

Let $\mathcal{A} = (A_1, \ldots, A_r)$ be a presentation of M
For $I \subseteq [r]$, let

$$\mathcal{A}^I = \begin{cases} A_i \cup x, & \text{if } i \in I, \\ A_i, & \text{otherwise.} \end{cases}$$

The matroid $M[\mathcal{A}^I]$ is a transversal extension of M

Notation

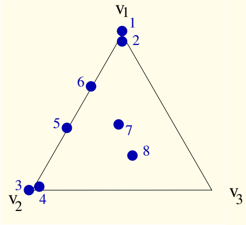
Let $\mathcal{A} = (A_1, \ldots, A_r)$ be a presentation of M
For $I \subseteq [r]$, let

$$\mathcal{A}^I = \begin{cases} A_i \cup x, & \text{if } i \in I, \\ A_i, & \text{otherwise.} \end{cases}$$

The matroid $M[\mathcal{A}^I]$ is a transversal extension of M

Example: $\mathcal{A} = (\{1,2,5,6,7,8\}, \{3,4,5,6,7\}, \{7,8\})$

$I = \{1,2,3\}$



Notation

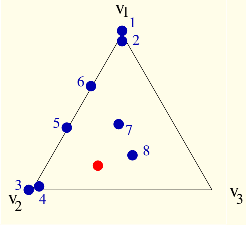
Let $\mathcal{A} = (A_1, \ldots, A_r)$ be a presentation of M
For $I \subseteq [r]$, let

$$\mathcal{A}^I = \begin{cases} A_i \cup x, & \text{if } i \in I, \\ A_i, & \text{otherwise.} \end{cases}$$

The matroid $M[\mathcal{A}^I]$ is a transversal extension of M

Example: $\mathcal{A} = (\{1,2,5,6,7,8\}, \{3,4,5,6,7\}, \{7,8\})$

$I = \{1,2,3\}$



Notation

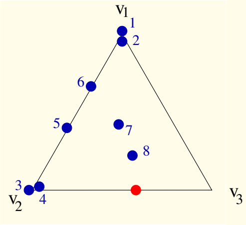
Let $\mathcal{A} = (A_1, \ldots, A_r)$ be a presentation of M
For $I \subseteq [r]$, let

$$\mathcal{A}^I = \begin{cases} A_i \cup x, & \text{if } i \in I, \\ A_i, & \text{otherwise.} \end{cases}$$

The matroid $M[\mathcal{A}^I]$ is a transversal extension of M

Example: $\mathcal{A} = (\{1,2,5,6,7,8\}, \{3,4,5,6,7\}, \{7,8\})$

$I = \{2,3\}$



Notation

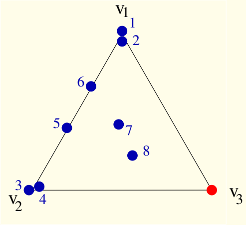
Let $\mathcal{A} = (A_1, \ldots, A_r)$ be a presentation of M
For $I \subseteq [r]$, let

$$\mathcal{A}^I = \begin{cases} A_i \cup x, & \text{if } i \in I, \\ A_i, & \text{otherwise.} \end{cases}$$

The matroid $M[\mathcal{A}^I]$ is a transversal extension of M

Example: $\mathcal{A} = (\{1,2,5,6,7,8\}, \{3,4,5,6,7\}, \{7,8\})$

$I = \{3\}$



Notation

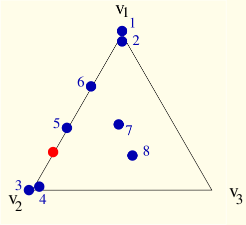
Let $\mathcal{A} = (A_1, \ldots, A_r)$ be a presentation of M
For $I \subseteq [r]$, let

$$\mathcal{A}^I = \begin{cases} A_i \cup x, & \text{if } i \in I, \\ A_i, & \text{otherwise.} \end{cases}$$

The matroid $M[\mathcal{A}^I]$ is a transversal extension of M

Example: $\mathcal{A} = (\{1,2,5,6,7,8\}, \{3,4,5,6,7\}, \{7,8\})$

$I = \{1, 2\}$



Only minimal presentations give different extensions

Thm 1

The following are equivalent:

(i) if $I \neq J$ then $M[\mathcal{A}^I] \neq M[\mathcal{A}^J]$

(ii) the presentation $\mathcal{A}$ is minimal

Cor If $\mathcal{A}$ is a minimal presentation of M, then $\mathcal{A}^I$ is a minimal presentation of $M[\mathcal{A}^I]$

Minimal presentations give all possible extensions

Thm 2

If N is a transversal extension of M, there exist a minimal presentation $\mathcal{A}$ of M and a set $I \subseteq [r]$ such that $N = M[\mathcal{A}^I]$

Minimal presentations give all possible extensions

Thm 2

If N is a transversal extension of M, there exist a minimal presentation $\mathcal{A}$ of M and a set $I \subseteq [r]$ such that $N = M[\mathcal{A}^I]$

Unfortunately we do not know how to get all minimal presentations of all extensions...

Proof of Thm 1 (I)

Thm 1 The following are equivalent:

(i) if $I \neq J$ then $M[\mathcal{A}^I] \neq M[\mathcal{A}^J]$

(ii) the presentation $\mathcal{A}$ is minimal

Proof of (ii) $\Rightarrow$ (i)

Proof of Thm 1 (I)

Thm 1 The following are equivalent:

(i) if $I \neq J$ then $M[\mathcal{A}^I] \neq M[\mathcal{A}^J]$

(ii) the presentation $\mathcal{A}$ is minimal

Proof of (ii) $\Rightarrow$ (i)

As $\mathcal{A}$ is minimal, the A_i are cocircuits, so $H_i = E - A_i$ are hyperplanes of $M[\mathcal{A}]$:

Proof of Thm 1 (I)

Thm 1 The following are equivalent:

(i) if $I \neq J$ then $M[\mathcal{A}^I] \neq M[\mathcal{A}^J]$

(ii) the presentation $\mathcal{A}$ is minimal

Proof of (ii) $\Rightarrow$ (i)

As $\mathcal{A}$ is minimal, the A_i are cocircuits, so $H_i = E - A_i$ are hyperplanes of $M[\mathcal{A}]$:

- If $i \in I$, then H_i is a hyperplane of $M[\mathcal{A}^I]$

Proof of Thm 1 (I)

Thm 1 The following are equivalent:

(i) if $I \neq J$ then $M[\mathcal{A}^I] \neq M[\mathcal{A}^J]$

(ii) the presentation $\mathcal{A}$ is minimal

Proof of (ii) $\Rightarrow$ (i)

As $\mathcal{A}$ is minimal, the A_i are cocircuits, so $H_i = E - A_i$ are hyperplanes of $M[\mathcal{A}]$:

- If $i \in I$, then H_i is a hyperplane of $M[\mathcal{A}^I]$

- If $i \notin I$, then $H_i \cup x$ is a hyperplane of $M[\mathcal{A}^I]$

Proof of Thm 1 (I)

Thm 1 The following are equivalent:

(i) if $I \neq J$ then $M[\mathcal{A}^I] \neq M[\mathcal{A}^J]$

(ii) the presentation $\mathcal{A}$ is minimal

Proof of (ii) $\Rightarrow$ (i)

As $\mathcal{A}$ is minimal, the A_i are cocircuits, so $H_i = E - A_i$ are hyperplanes of $M[\mathcal{A}]$:

- If $i \in I$, then H_i is a hyperplane of $M[\mathcal{A}^I]$

- If $i \notin I$, then $H_i \cup x$ is a hyperplane of $M[\mathcal{A}^I]$

So looking which H_i are hyperplanes in $M[\mathcal{A}^I]$ we can recover I

Proof of Thm 1 (II)

Thm 1 The following are equivalent:

(i) if $I \neq J$ then $M[\mathcal{A}^I] \neq M[\mathcal{A}^J]$

(ii) the presentation $\mathcal{A}$ is minimal

Idea of proof of (i) $\Rightarrow$ (ii)

Proof of Thm 1 (II)

Thm 1 The following are equivalent:

(i) if $I \neq J$ then $M[\mathcal{A}^I] \neq M[\mathcal{A}^J]$

(ii) the presentation $\mathcal{A}$ is minimal

Idea of proof of (i) $\Rightarrow$ (ii)

Assume $\mathcal{A}$ is not minimal. Say A_r is not a cocircuit

Proof of Thm 1 (II)

Thm 1 The following are equivalent:

(i) if $I \neq J$ then $M[\mathcal{A}^I] \neq M[\mathcal{A}^J]$

(ii) the presentation $\mathcal{A}$ is minimal

Idea of proof of (i) $\Rightarrow$ (ii)

Assume $\mathcal{A}$ is not minimal. Say A_r is not a cocircuit

We claim that $M[\mathcal{A}^{[r-1]}] = M[\mathcal{A}^{[r]}]$

Proof of Thm 1 (II)

Thm 1 The following are equivalent:

(i) if $I \neq J$ then $M[\mathcal{A}^I] \neq M[\mathcal{A}^J]$

(ii) the presentation $\mathcal{A}$ is minimal

Idea of proof of (i) $\Rightarrow$ (ii)

Assume $\mathcal{A}$ is not minimal. Say A_r is not a cocircuit

We claim that $M[\mathcal{A}^{[r-1]}] = M[\mathcal{A}^{[r]}]$

It is essentially a consequence of the lemma of Bondy and Welsh

The weak order

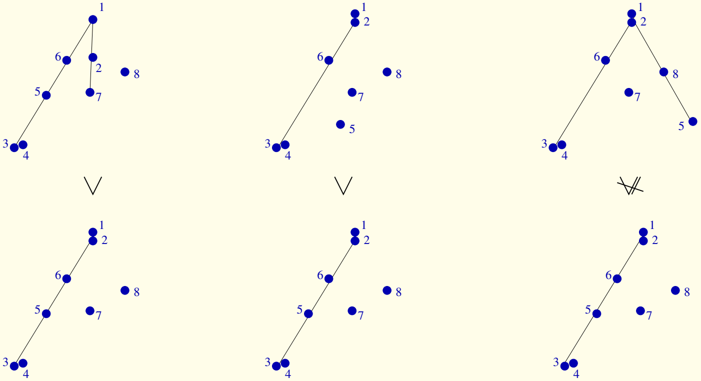
Let M_1, M_2 be two matroids on E. The weak order:

$M_1 \leq_w M_2$ if every independent set in M_1 is independent in M_2

The weak order

Let M_1, M_2 be two matroids on E. The weak order:

$M_1 \leq_w M_2$ if every independent set in M_1 is independent in M_2



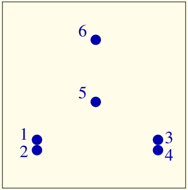
Extensions and the weak order

Fact: the set of all extensions of a matroid M is a lattice under the weak order

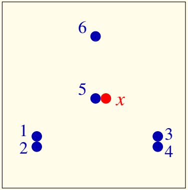
Extensions and the weak order

Fact: the set of all extensions of a matroid M is a lattice under the weak order

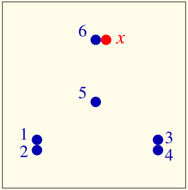
M



M_5

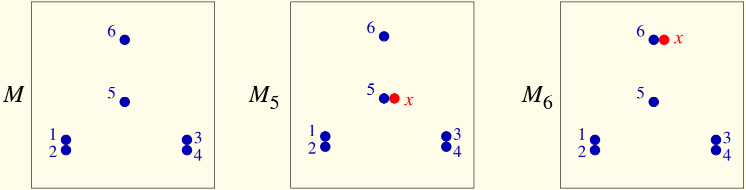


M_6

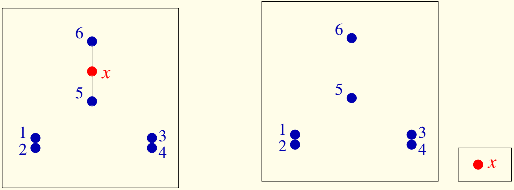


Extensions and the weak order

Fact: the set of all extensions of a matroid M is a lattice under the weak order



Join $M_5 \vee M_6$ and meet $M_5 \wedge M_6$:

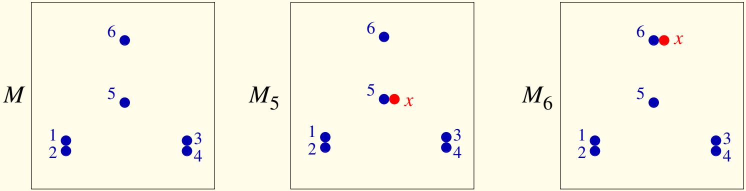


A question

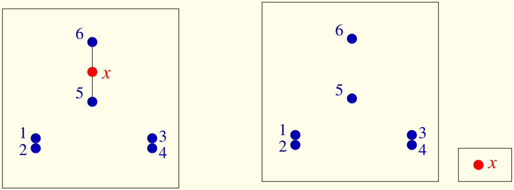
Is the set of all transversal extensions of a transversal matroid also a lattice under the weak order?

- Thus, given N_1 and N_2 two transversal extensions of M, is there a smallest transversal extension N_3 such that $N_1 \leq_w N_3$ and $N_2 \leq_w N_3$?

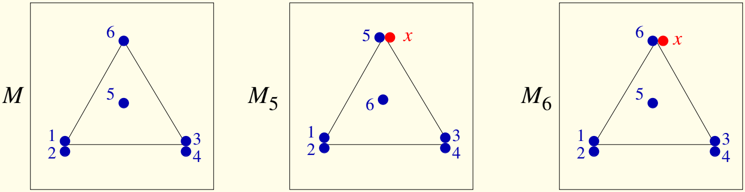
Example



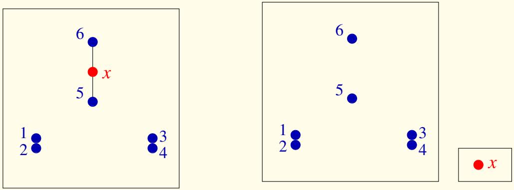
Join $M_5 \vee M_6$ and meet $M_5 \wedge M_6$:



Example



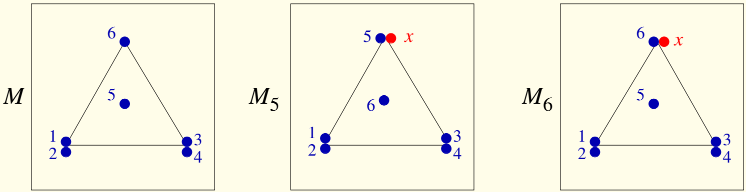
Join $M_5 \vee M_6$ and meet $M_5 \wedge M_6$:



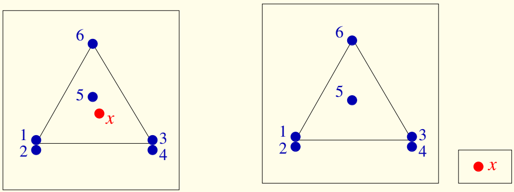
M_5 and M_6 are transversal extensions of M

But the ordinary join $M_5 \vee M_6$ is not transversal

Example



Transversal join $M_5 \vee M_6$ and meet $M_5 \wedge M_6$:



M_5 and M_6 are transversal extensions of M

But the ordinary join $M_5 \vee M_6$ is not transversal

One (predictable?) result

Thm

For a transversal matroid $M[\mathcal{A}]$, the set of extensions obtained by adding a new element x to some of the sets in $\mathcal{A}$ is a lattice under the weak order.

The transversal join of $M[\mathcal{A}^I]$ and $M[\mathcal{A}^J]$ is $M[\mathcal{A}^{I \cup J}]$

Yet their transversal meet need not be $M[\mathcal{A}^{I \cap J}]$ if $\mathcal{A}$ is not minimal

Wrapping up

- We know how to get all transversal extensions of a transversal matroid
  We know some ways in which repetitions can arise

Wrapping up

- We know how to get all transversal extensions of a transversal matroid
  We know some ways in which repetitions can arise

  We would like to get each extension only once!

Wrapping up

- We know how to get all transversal extensions of a transversal matroid
  We know some ways in which repetitions can arise

  We would like to get each extension only once!

- We understand a little the structure of the set of transversal extensions

Wrapping up

- We know how to get all transversal extensions of a transversal matroid
  We know some ways in which repetitions can arise

  We would like to get each extension only once!

- We understand a little the structure of the set of transversal extensions

  We would like to understand it better!

Wrapping up

- We know how to get all transversal extensions of a transversal matroid
  We know some ways in which repetitions can arise

  We would like to get each extension only once!

- We understand a little the structure of the set of transversal extensions

  We would like to understand it better!

- We would like to use our results!